ASYMPTOTIC PROPERTIES OF SOLUTIONS FOR SOME FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

BY

AHMAD MAHADI MUGBIL AHMAD

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This dissertation, written by **AHMAD MAHADI MUGBIL AHMAD** under the direction of his dissertation adviser and approved by his dissertation committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS**.

Dissertation Committee

Prof. Nasser-eddine M. Tatar (Adviser)

Prof. Khaled M. Furati (Co-adviser)

Prof. Salim A. Messaoudi (Member)

Prof. Mohamed A. El-Gebeily (Member)

Kassem Mustapha

Prof. Kassem A. Mustapha (Member)

Dr. Hussain S. Al-Attas
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

عمادة الدراسات العليا
DEANSHIP OF GRADUATE STUDIES

17/5/16

Date

To my parents, wife, children, sister and brothers who waited patiently for me to come out of this study.
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THESIS ABSTRACT

NAME: Ahmad Mahadi Mugbil Ahmad

TITLE OF STUDY: Asymptotic Properties of Solutions for Some Fractional Integro-differential Equations

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There are many results in the literature on the asymptotic behavior of ordinary differential equations. Under certain conditions, solutions of some ordinary differential equations are asymptotic to lines or in general to polynomials. In contrast, there are very few studies on the asymptotic behavior of solutions fractional integro-differential equations. In this dissertation we would like to contribute in filling this gap by studying the asymptotic (long-time) behavior of solutions for some general classes of fractional integro-differential equations. There exist different kinds of fractional derivatives in the literature. We consider the most commonly used ones: the Riemann-Liouville and Caputo fractional derivatives. Reasonable sufficient conditions under which the solutions behave like power functions at infinity are established. For this purpose, we combine and generalize some

well-known integral inequalities with some desingularization techniques.

The nonexistence of solutions of some classes of fractional integro-differential inequalities is investigated. The Caputo and Riemann-Liouville fractional derivatives of sub-first and sub-second orders are treated both. The nonlinear source term consists of a convolution of a (possibly singular) kernel with a polynomial of the state. We establish various criteria under which there are no (nontrivial) global solutions. To that end, the test function method to the weak formulation of the problem is applied. Then, we use suitable estimation techniques and inequalities to achieve our objectives. Some examples are provided to illustrate our findings.

Our results could be utilized to identify the limitations of many physical systems and to analyze the behavior of solutions of some nonlinear fractional differential equations and inequalities for which the explicit solution may not be available. Also our results will extend the abundant results on integer-order problems to the (limited results available for) fractional-order problems.

ملخص الرسالة

الاسم : أحمد مهدي مقبل أحمد

عنوان الرسالة : الخصائص التقاربية لحلول بعض المعادلات التفاضلية التكاملية الكسورية

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هناك العديد من النتائج المتوفرة حول السلوك التقاربي للمعادلات التفاضلية العادية. وفقا لاستقصائنا, هناك عدد قليل جدا من الدراسات التي تتناول السلوك التقاربي لحلول المعادلات التفاضلية التكاملية الكسورية (ذات رتب غير صحيحة).

في هذه الأطروحة ملأنا هذه الفجوة من خلال دراسة السلوك التقاربي (على المدى البعيد) لحلول نوع عام من المعادلات التفاضلية التكاملية الكسورية. هناك العديد من المشتقات الكسورية, غير أننا تناولنا أكثرها شيوعا: مشتقات ريمان-ليوفيل وكابوتو الكسورية. تم تحديد شروطا كافية و معقولة بموجبها تتصرف الحلول عند قيم الزمن الكبيرة كدوال قوى. لهذا الغرض، قمنا بدمج و تطوير بعض المتراجحات التكاملية مع بعض تقنيات إزالة عدم الانتظام.

كما قمنا بدراسة عدم وجود حلول لبعض المتراجحات التفاضلية التكاملية الكسورية, حيث تعاملنا مع مشتقات كابوتو وريمان-يوفيل من رتب متعددة. لقد حددنا معايير مختلفة لا توجد بموجبها حلول شاملة غيرصفرية. تحقيقا لهذه الغاية، تم تطبيق طريقة الدالة الاختبارية للصيغة الضعيفة للمشكلة. ثم بعد ذلك، استخدمنا تقنيات تقدير ومتراجحات مناسبة لتحقيق أهدافنا. عدة أمثلة قمنا بضربها لشرح النتائج التي توصلنا إليها.

يمكن استخدام النتائج التي توصلنا إليها لتحديد أوجه القصور في العديد من النظم الفيزيائية وتحليل سلوك حلول المعادلات و المتراجحات التفاضلية الكسورية غير الخطية التي لا تمتلك حلول صريحة. أيضا نتائجنا توسع و تعمم النتائج المتوفرة للعديد من المسائل ذات الرتب الصحيحة و حتى تلك القليلة المتوفرة للرتب غير الصحيحة.

CHAPTER 1

INTRODUCTION

In this chapter we present the motivation and outline of our investigation. A brief history of fractional calculus and some recent areas of applications of fractional differential equations are given in Section 1.1. A short overview of the long-time behavior of solutions of differential equations is introduced in Section 1.2. In Section 1.3, we state the problems of study. The objectives of our research are listed in Section 1.4. In Section 1.5, we explain our approaches to achieve the research objectives. The significance of the study of the asymptotic properties of solutions for fractional integro-differential equations is discussed in Section 1.6. We highlight our main contributions in Section 1.7 and give an outline of the remaining chapters in Section 1.8.

1.1 Fractional Calculus

Fractional calculus is the generalization of the classical Newtonian calculus to arbitrary orders. It deals with the generalizations of differentiation and integration to any order, not necessarily integer. The fractional derivatives and integrals are not local properties. Thereby they take into account the history of the system and the nonlocal effects. So it seems that fractional calculus interprets the reality of nature better.

The theory of fractional derivatives goes back to Leibniz's notes, in which the meaning of the derivative of order $\frac{1}{2}$ is discussed. L'Hopital wrote a letter to Leibniz dated 30th September 1695, asking what would be the result if the order $n = \frac{1}{2}$ in particular notation Leibniz has used for the nth derivative of a function $\frac{d^n f(x)}{dx^n} = D^n f(x)$, where n is a positive integer. Leibniz replied "It will lead to a paradox. From this apparent paradox, one day useful consequences will be drawn." Indeed these few words announced the birth of fractional calculus. Studies over the last three hundred years have proved Leibniz's prediction. Later the question turned out to be: Can n be any fractional, real or complex number? The answer of this question was affirmative but the name still fractional order derivatives and it might be better to called them arbitrary order derivatives. During the time, foundations of fractional calculus has been established by many famous mathematicians, e.g. Grunwald, Liouville, Riemann, Euler, Abel, Lagrange, Heaviside, Fourier etc. Many of them set original approaches, which outlined in [1, 2].

In fractional calculus, the fractional derivatives are defined through fractional integrals. Several different derivatives have been defined and studied: Grunwald-Letnikov, Riemann-Liouville, Caputo, Hadamard and Erdélyi–Kober are just a few to name [3, 4, 2]. A recent generalization of the Riemann-Liouville and Hadamard fractional operators introduced by Katugampola in 2011 [5, 6]. In this dissertation we consider only the most common derivatives, Riemann-Liouville and Caputo which will be presented in Chapter 3. Some major contributions and events in fractional calculus during the last fifty years are reported in [7, 8, 9, 10].

For three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics. Many books were written on the theories and developments of fractional calculus [11, 12, 13, 3, 14, 15, 16, 4, 2, 17, 18, 19]. In [3] and [2], Kilbas *et al.* and Samko *et al.*, respectively, provided a comprehensive study of the subject.

In the last few decades, the theory of fractional differential equations turned out to be very attractive not only to mathematicians but also to physicists, engineers, biologists, and economists. Numerous phenomena in various fields of science and engineering can be described by fractional differential and integro-differential equations. The differential and integro-differential equations of noninteger orders describe many systems better than those of integer orders especially while studying hereditary phenomena and systems with memory. A considerable interest in the theory and applications of fractional differential equations is witnessed in [20, 21, 22, 3, 14, 23, 2, 24]. Some recent applications include viscoelasticity [25, 26, 27, 28, 29], mechanics [30, 31, 32, 33, 34, 35], electromagnetic theory [36, 37], rheology [38, 39], polymer physics [40], control systems [41, 42, 43, 44], bioengineering [45, 46], digital circuit synthesis [47], robots, nanotechnology [48, 20, 49], and stochastic processes [50, 51, 52, 53, 54] and so many others [55, 56, 57, 58, 22, 59, 60, 61, 62].

1.2 Long-time Behavior

Analyzing the asymptotic behavior of functions is a mathematical analytic tool that describes limiting behavior. It is a main tool for studying the ordinary and partial differential equations which arise in the mathematical models of real life phenomena. Indeed, the asymptotic behavior analysis is used to determine the asymptotic behavior of solutions to a differential equation without fully solving it. The asymptotic behavior has many applications in different scientific fields. It is used in applied mathematics to construct numerical methods to approximate solutions. In the theoretical physics it is used to analyze the behavior of physical systems when they are very large. In computer science, the asymptotic behavior is used in analyzing the performance of algorithms when applied to very huge input data sets.

Several aspects of the qualitative theory of differential equations and dynamical systems deal with asymptotic properties of solutions. These properties are about what happens with the system after a long period of time.

1.3 Problem Statement

We consider the following class of nonlinear fractional integro-differential equations

$$\left(D^{\mu}_{0+}x\right)(t) = f\left(t, (D^{\beta}_{0+}x)(t), \int_0^t k(t, s, \left(D^{\gamma}_{0+}x\right)(s))ds\right), \; t > 0, \tag{1.1}$$

supplied with appropriate initial data. The derivatives D^{μ}_{0+}, D^{β}_{0+} and D^{γ}_{0+} represent either the Caputo or the Riemann-Liouville fractional derivatives of orders μ, β and γ, respectively, where $0 \leq \beta < \mu$ and $0 \leq \gamma < \mu$. The definitions of the Caputo and the Riemann-Liouville fractional derivatives are given in Chapter 3, Section 3.2.

1.4 Research Objectives

1. We study the asymptotic behavior of solutions for the nonlinear fractional integro-differential equation (1.1) when the fractional derivative is of Caputo type and Riemann-Liouville type and $1 < \mu < 2$. More precisely, we achieved the following:

   (a) Studying different types of the nonlinear function f and the kernel k. In this regard, we considered the following cases:

   - Cases of fractional and non-fractional source terms.

   - Cases of convolution and non-convolution kernels.

- Cases of bounded kernels.

- Case of singular kernel.

- Case of delay.

(b) Finding an appropriate underlying space.

(c) Making use of some known and ad-hoc inequalities.

(d) Determining sufficient conditions on the different nonlinearities.

(e) Illustrating our results as much as possible with examples.

2. Under some sufficient conditions, we prove some boundedness results for solutions of (1.1) when the fractional derivative is of Caputo type and $0 < \mu < 1$.

3. We study the power-type decay of solutions for the nonlinear fractional integro-differential equation (1.1) when the fractional derivative is of Riemann-Liouville type and $0 < \mu < 1$.

4. We discuss nonexistence of (nontrivial) global solutions for the initial value problem formed by (1.1) subject to some appropriate initial conditions when $0 < \mu < 1$ and $1 < \mu < 2$. The function f in the right-hand side of (1.1) satisfies

$$
\begin{aligned}
& f\left(t, (D_{0^+}^{\beta} x)(t), \int_0^t k(t, s, \left(D_{0^+}^{\gamma} x\right)(s)) ds\right) \\
& \geq -\sigma (D_{0^+}^{\beta} x)(t) + \int_0^t h(t-s) \left|x(s)\right|^q ds,
\end{aligned}
$$

for some nonnegative locally integrable function h, $\sigma = 0, 1$. We establish some criteria under which there are no (nontrivial) global solutions. These criteria are in the form of sufficient conditions on the kernel h, the orders, the exponent q, and the initial conditions. In this regard, we

- find an appropriate underlying space,

- determine a suitable test function,

- find the weak formulation of the problem,

- set sufficient conditions on the different parameters,

- consider general forms of the kernel in the convolution term,

- discuss how the (integer or fractional) damping affects the range of non-existence,

- illustrate our results with examples and special well-known equations.

1.5 Methodology

The main procedures and techniques we used to achieve our objectives include:

- Fractional calculus theory and tools to treat the terms involving fractional integrals and derivatives.

- Considering the associated Volterra integral equations corresponding to (1.1).

- Using the properties of fractional derivatives to find appropriate underlying spaces for the solutions based on specific equations or inequalities.

- Test function method to establish non-existence results.

- Desingularization techniques to estimate the singular terms.

- Making use of some crucial estimations.

- Making use of some known or ad-hoc inequalities.

- Making use of generalized versions of Bihari-type inequalities and comparison tests to obtain bounds.

- Selecting the appropriate initial conditions based on the fractional derivative type.

- Making use of asymptotic theories to analyze the long run behavior of the solutions.

1.6 Significance of Study

From both the theoretical point of view and the application point of view, it is of great importance to have an idea about the behavior of solutions for large values of the time variable. The significance of studying the asymptotic properties for solutions of differential and integral equations is well recognized especially when these equations cannot be solved explicitly. The study of asymptotically linear solutions to linear and nonlinear ordinary differential equations is important in many fields like fluid mechanics, differential geometry, bidimensional gravity, Jacobi fields, etc. see e.g. [63].

The existence of asymptotically linear solutions is related to many analytic properties like existence of bounded, monotonic, non-oscillatory, square integrable solutions and eventually positive (negative) solutions. As a consequence, the topic of long-time behavior of solutions for ordinary differential equations attracted many researchers, e.g. see [64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83].

The study of the asymptotic behavior of the solutions is not well-developed in the theory of fractional differential equations and fractional integro-differential equations. Indeed, we can find only very few papers [84, 85, 86, 87, 88, 89, 90, 91, 92, 93] in the literature dealing with special kinds of problems.

It is known from the definitions of fractional derivatives that they use in someway all the history of the state through a convolution with a singular kernel. Moreover, in the case of fractional integro-differential equations, the source term may involve additional singularities in the kernel. Because of all these issues, it is difficult to apply the existing approaches and methods in the literature for integer order to the noninteger.

As is well known, studying nonexistence of solutions for differential equations is as important as studying the existence of solutions. The sufficient conditions of nonexistence of solutions provide necessary conditions for existence of solutions. Investigating the nonexistence of solutions for differential equations provide very important and necessary information on limiting behaviors of many physical systems. It is also interesting to know what could happen to these solutions such as blowing up in finite time or at infinity. In industry, knowing the blow up in finite time can prevent accidents and malfunction. It helps also improving the performance of machines and extending their life-span.

1.7 Main Contribution

We establish useful results for analyzing the long-time behavior of many history dependent systems. Reasonable sufficient conditions ensuring that no (nontrivial) solutions can exist for all time, are determined. Some nonexistence criteria when the fractional derivatives are of Caputo type or Riemann-Liouville type are found. We generalize and extend the relevant existing results for integer orders and provide new benchmarking and study cases to the community of computing and analysis.

To the best of our knowledge, there are no similar investigations on the long-time behavior and nonexistence of solutions for fractional integro-differential equations and inequalities of type (1.1). The situation is different from the integer order. The few works on nonexistence of solutions, we aware of, are concerned with a polynomial type source. This is of course a special case corresponding to the Dirac delta function as kernel in our proposed nonlocal problem.

Our results could be used to analyze the long-time behavior of solutions for the nonlinear fractional integro-differential problems which have no explicit solutions. Such problems arise in many areas such as: optimal control, dynamic systems, mechanical structures, viscoelasticity. In the area of mathematical analysis, our results will extend the results for integer order problems to fractional order problems.

1.8 Dissertation Outline

This dissertation consists of seven chapters, including the current introduction chapter which introduces the research work, provides motivation and significance of study, states the problem, the research objectives, the research approach, and our main contribution.

Chapter 2 contains a review of previous works concerning existence, asymptotic behavior, long-time behavior, blow-up and nonexistence of solutions for differential and integro-differential equations of integer and noninteger orders.

In Chapter 3 we present the used notations, underlying function spaces, background material and some preliminary results. It contains, in particular, the definitions and basic properties of the fractional integrals and derivatives used in this dissertation. Some useful lemmas and inequalities that will be used later in our proofs are listed there.

The asymptotic behavior of solutions is studied in detail in Chapter 4. In this chapter, we consider the initial value problem formed by the nonlinear fractional integro-differential equation (1.1) of order $1 < \mu < 2$ subject to some appropriate initial conditions. Several sufficient conditions related to different cases of nonlinearities are handled.

Chapter 5 is concerned with the long-time behavior of solutions when $0 < \mu < 1$. The boundedness of solutions is investigated when the fractional derivative is of Caputo type. For the Riemann-Liouville fractional derivative, the power-type decay is discussed in this chapter.

In Chapter 6 the problem of nonexistence of nontrivial global solutions is tackled in a weighted space of continuous functions by the method of test functions. Different nonexistence criteria are established in the form of sufficient conditions on the kernel, the derivative, the orders, the exponent, and the initial conditions. Conclusions and possible directions for future works are briefly discussed in the last chapter, Chapter 7.

CHAPTER 2

LITERATURE REVIEW

This chapter is devoted to a review of the literature. We review works concerning existence, nonexistence, blow-up, decay and asymptotic behavior of solutions of some differential and integro-differential equations of integer and noninteger orders. Section 2.1 contains results on existence of solutions for some classes of fractional integro-differential equations. In Section 2.2, we survey recent results on the long-time behavior of solutions for several classes of differential, integro-differential, fractional and fractional integro-differential equations. Section 2.3 is devoted to some nonexistence and blow-up results existing in the literature.

2.1 Existence Results for Fractional Integrodifferential Equations

There is a great volume of literature on existence of solutions for various classes of fractional differential and integro-differential equations, (e.g. see [94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113]). Agarwal, *et al.* surveyed in [94] many of these existence results. They focused on initial and boundary value problems for fractional differential equations with Caputo fractional derivatives of orders between 0 and 1 and between 1 and 2.

Furati and Tatar considered in [103], the Cauchy-type fractional differential problem

$$\begin{cases} \left(D_{0^+}^{\alpha}x\right)(t) = f(t, x(t)) + \int_0^t k(t, s, x(s))ds, \ t > 0, \\ \lim\limits_{t \to 0+} t^{1-\alpha}x(0^+) = b, \ b \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, $0 < \alpha < 1$. They used the Schauder fixed point theorem to prove a local existence result in the $C_{1-\alpha}[0, \infty)$ (see Definition 3.6 in Section 3.1) for some classes of the nonlinearities f and k involving some power functions on t, s and x. The case of $k \equiv 0$ has been studied by Delbosco and Rodino [114], Kilbas *et al.* [115], and many others.

In their paper [108], Kirane, Medved, and Tatar, considered the second-order semilinear Volterra integro-differential problem

$$\begin{cases} x''(t) = Ax(t) + f(t) + \int_a^t g\left(t, s, x(s), \left(D_{0^+}^{\beta_1}x\right)(s), ..., \left(D_{0^+}^{\beta_n}x\right)(s)\right)ds, \ t > 0, \\ x(0) = x_0, \ x'(0) = x_1, \quad x_0, \ x_1 \in X, \end{cases}$$

where the fractional derivatives $D_{0^+}^{\beta i}$, $0 < \beta_i \leq 1$, $i = 1, ..., n$, are of Riemann-Liouville or Caputo type. Here A is an operator that generates a strongly continuous cosine family of bounded linear operators in the Banach space X. The nonlinear function $f : \mathbb{R}_+ = [0, \infty) \to X$ is assumed to be continuously differentiable and the nonlinear function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times X \times ... \times X \to X$ is assumed to be continuous, continuously differentiable with respect to its first variable and Lipschitz continuous with respect to the other variables.

They proved existence and uniqueness of the solutions in the space of continuously differentiable functions in case of Caputo fractional derivatives. When the fractional derivatives $D_{0^+}^{\beta i}$ are of Riemann-Liouville type, they also required the continuity of these derivatives.

In 2012, Trujillo *et al.* [98] established the existence and uniqueness of solutions for the nonlinear fractional integro-differential problem

$$\begin{cases} \left({}^C D_{a^+}^{\alpha} x\right)(t) = f\left(t, \left({}^C D_{a^+}^{\beta} x\right)(t), \int_a^t g\left(t, s, x(s)\right) ds\right), \ t \in (a, b], \\ x^{(k)}(a) = c_k, \ k = 0, 1, ..., m-1, \end{cases}$$

where ${}^C D_{a^+}^{\alpha}$ is the Caputo fractional derivative of order α, $m - 1 < \alpha < m$, $n-1 < \beta < n$, $\beta < \alpha$, $m, n \in \mathbb{N}$, $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

They showed that this problem has a unique solution $x \in C^{m-1}[a, b]$ with ${}^C D_{a^+}^{\alpha} x \in C[a, b]$. Their main tool is the fixed point theorem for nonself mappings. First, using a suitable substitution, they constructed an equivalent fractional integral equation. Then, used some fractional integral inequalities and the nonlinear alternative Leray-Schauder type to achieve their existence result. Moreover, under an additional condition, the uniqueness of solution was established by using the Banach contraction principle.

2.2 Asymptotic Behavior Results

The long-time behavior of solutions of differential equations attracted many researchers. In many cases, the main idea is to establish sufficient reasonable conditions ensuring comparison or similarity with the long-time behavior of solutions of simpler differential equations. This section is divided into three subsections. In the first subsection we present some results in the literature concerning the asymptotic behavior of solutions for different forms of integer order differential equations. Subsection 2.1.2 contains similar results for fractional differential equations. The few available works on the long-time behavior of solutions for some fractional integro-differential equations are indicated in Subsection 2.1.3.

2.2.1 Asymptotic Behavior of Solutions for Differential Equations of Integer Order

The asymptotic behavior of solutions for ordinary second order differential equations has been studied by many authors, e. g. Bihari [65], Cohen [66], Constantin [67], Tong [116], Trench [117], Naito *[118],* Kusano, *et al.* [72, 71], Rogovchenko [119], Rogovchenko, *et al.* [120], Philos [121], Philos, *et al.* [122] and [123], Pinto [124], Pachpatte [125], Coppel [68], Agarwal, *et al.* [64], Mustafa [78, 79], Mustafa, *et al.* [93, 80, 81, 82, 83].

In 1957, using his generalization of Bellman Lemma, Bihari [65] studied the second order nonlinear differential equation

$$x''(t) + f(t)g(x(t)) = 0,\ t \geq 1,$$

and showed that $\lim\limits_{t\to\infty} x'(t)$ exists, provided that $f : [1, \infty) \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying

$$\int_1^\infty t\,|f(t)|\,dt < \infty \quad \text{and} \quad \frac{|g(y)|}{t} \leq h\left(\frac{|y|}{t}\right) \text{ for all } y,\ t \geq 1,$$

where $h : [0, \infty) \to [0, \infty)$ is nondecreasing, continuous function with $h(0) = 0$, $h(y) > 0$ for all $y > 0$ and $\int_0^\infty \frac{t}{h(t)} dt = \infty$.

Cohen [66] proved in 1976 that the nonlinear differential equation

$$x''(t) = f(t, x(t)),\ t \geq 1, \tag{2.2}$$

has a solution asymptotic to $a + bt$ at infinity, where a, b are real constants, $b \neq 0$. The function f in (2.2) is assumed to satisfy the following conditions:

1. $f(t, x)$ is continuous in $D = \{(t, x) : t \geq 1,\ x \in \mathbb{R}\}$,
2. $\frac{\partial f}{\partial x}$ exists and satisfies $\frac{\partial f(t,x)}{\partial x} > 0$ on D,
3. $|f(t, x)| \leq \frac{\partial f(t,0)}{\partial x} |x(t)|$ on D,
4. $\int_1^\infty t \frac{\partial f(t,0)}{\partial x} dt < \infty$.

The asymptotic behavior of solutions for (2.2) has been the subject of intensive research and studied under different conditions by many other authors see e.g. Tong [116] (1982), Kusano and Trench [72], [71] (1985), Rogovchenko [119] (1998), Constantin [126] (2005), Mustafa and Rogovchenko [80] and [81] (2006).

In 1985, Kusano and Trench [71] established some conditions ensuring that the solutions of the nonlinear differential equation

$$x^{(n)} + f(t,x) = 0,\ t \geq t_0 \geq 0, \tag{E}$$

exist on $[t_0, \infty)$ and have the asymptotic behavior

$$\lim_{t\to\infty} t^{-k}x(t) = c \neq 0.$$

These conditions are explicitly given in the following theorem:

Theorem *Suppose that f is continuous and satisfies an inequality of the form*

$$|f(t,x)| \leq \phi(t)F(x) \text{ for } (t,x) \in [t_0,\infty) \times (0,\infty),\ t_0 \geq 0,$$

where $\phi : [t_0,\infty) \to [0,\infty)$ and $F : (0,\infty) \to [0,\infty)$ are continuous, $F(1) = 1$ and

$$\frac{1}{(n-k-1)!}\int_{t_0}^{\infty} t^{n-k-1}\phi(t)F(t^k)dt = M < \infty,\ k \in \{0,1,...,n-1\}.$$

Suppose also that F satisfies one of the following conditions:

(C_1) F is non-decreasing and $\lim_{t\to 0^+} F(x)/x = 0,$

(C_2) F is non-decreasing and $\lim_{t\to\infty} F(x)/x = 0$,

(C_3) F is non-increasing.

In addition, if $F(xz) \le F(x)F(z)$ for $x, z > 0$, θ and c are positive constants with $0 < \theta < 1$, the equation (E) has a solution x_0 on $[t_0, \infty)$ which belongs to

$$X := \left\{x \in C[t_0, \infty) : \left|x(t) - ct^k\right| \le c\theta t^k,\ t \ge t_0,\ 1 \le k \le n-1\right\},$$

and satisfies $\lim_{t\to\infty} t^{-k}x_0(t) = c \neq 0$, *provided c is sufficiently small if (C_1) holds, or sufficiently large if (C_2) or (C_3) holds.*

In 2004, Philos, *et al.* [122], studied solutions that behave asymptotically at infinity like polynomials of degree at most $n-1$ for the nth order ($n > 1$) nonlinear ordinary differential equation

$$x^{(n)}(t) = f(t, x(t)),\ t \ge t_0 > 0, \qquad (\mathrm{E}_1)$$

where f is a continuous real-valued function on $[t_0, \infty) \times \mathbb{R}$. Their main result is formulated in the following theorem:

Theorem *Let m be an integer with $1 \le m \le n-1$, and assume that*

$$|f(t,z)| \le p(t)g\left(\frac{|z|}{t^m}\right) + q(t) \quad \text{for all } (t,z) \in [t_0, \infty) \times \mathbb{R},$$

where p and q are nonnegative continuous real-valued functions on $[t_0, \infty)$ such *that*

$$\int_{t_0}^{\infty} t^{n-1}p(t)dt < \infty, \quad \int_{t_0}^{\infty} t^{n-1}p(t)dt < \infty,$$

and g *is a nonnegative continuous real-valued function on* $[0,\infty)$ *which is not identically zero. Let* $a_0, a_1, ..., a_m$ *be real numbers and* $T \geq t_0$*, and assume that there exists a positive constant* C *so that*

$$\left[\int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!}p(s)ds\right] \sup\left\{g(z) : 0 \leq z \leq \frac{C}{T^m} + \sum_{i=0}^{m} \frac{|a_i|}{T^{m-i}}\right\}$$

$$+ \int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!}q(s)ds \leq C.$$

Then, the differential equation (E_1*) has a solution* x *on the interval* $[T,\infty)$*, which is asymptotic to the polynomial* $a_0 + a_1t + \cdots + a_mt_m$ *for* $t \to \infty$*, i.e.*

$$x(t) = a_0 + a_1t + \cdots + a_mt_m + o(1).$$

The asymptotic behavior of solutions for the problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \ t > 1, \\ x(1) = c_1, \ x'(1) = c_2, \ c_1, c_2 \in \mathbb{R}, \end{cases} \tag{2.3}$$

has been studied by Dannan [69] in 1985. He proved that any solution x of (2.3) is asymptotic to $a + bt + o(t)$ as $t \to \infty$, under the following conditions:

1. the function $f(t,u,v)$ is continuous on $D = \{(t,u,v) : t > 1, \ u, v \in \mathbb{R}\}$,

2. $|f(t,u,v)| \leq \phi(t)\, g\left(\frac{|u|}{t}\right) + \psi(t)\,|v|$ for $(t,u,v) \in D$, where $\phi(t)$ and $\psi(t)$

are nonnegative continuous functions on $[1, \infty)$,

3. $g(u)$ is a nonnegative, continuous, nondecreasing function on $[0, \infty)$, and satisfies

$$|g(\alpha u)| \leq \phi_1(\alpha)\, g(u),$$

for $\alpha \geq 1$, $u \geq 0$, where $\phi_1(\alpha) > 0$ is continuous for $\alpha > 1$,

4. $\int_1^\infty \psi(t)\, dt = \theta_1 < \infty$, $\int_1^\infty \phi(t)\, dt = \theta_2 < \infty$ and there exists $M \geq 1$ such that

$$\Psi(t) \int_1^\infty \phi(s) \frac{\phi_1(M\Psi(s))}{\Psi^2(s)} ds \leq M \int_1^\infty \frac{ds}{g(s)},$$

where $\Psi(t) = \exp\left[\int_1^t \psi(s)\, ds\right]$, such that $|c_1| + |c_2| \leq M$ and

$$\lim_{t\to\infty} \int_1^t f(s, x(s), x'(s))\, ds$$

always exists.

The problem (2.3) also has been studied by Constantin [67] (1993), Rogovchenko [119] (1998), Rogovchenko [120] (2000), Lipovan [127] (2003), Mustafa and Rogovchenko [83] (2002), [82] (2004) and others.

Levin [73] in 1963 considered the equation

$$x'(t) = -\int_0^t h(t-s)g(x(s))ds, \ t > 0,$$

and proved that if $h \geq 0$, $h' \leq 0$, $h'' \geq 0$, $h''' \leq 0$, $xg(x) > 0$ for $x \neq 0$, and

$\int_0^{\pm\infty} g(\xi)d\xi = \infty$, then $x \to 0$, $x' \to 0$, $x'' \to 0$ as $t \to \infty$.

Halanay in [70] studied the same equation and obtained the same results of Levin but using less restrictive conditions on h, namely $h(t) - \epsilon_0 e^{-\lambda t}$ defines a positive kernel $(t \geq 0)$, $h' \in L^1$, and $\lim_{t\to\infty} h(t)$ exists.

The global existence and uniqueness for the problem

$$x'(t) = f\left(t, x(t), \int_0^t h(t,s)x(s)ds\right),$$

under a generalized Lipschitz-type condition on f and for a continuous kernel $h(t,s)$ on $\Delta = \{(t,s) : 0 \leq s \leq t < T \leq \infty\}$ is proved in [128]. Well-posedness for a similar problem has been proved in [129] using Darbo's fixed point theorem for a smooth kernel and under some growth condition on the nonlinearity. When the kernel is summable over $(0,\infty)$ and its L^1-norm is small enough, the global existence of solution is established (see [129]).

In 2013, Cheng and Ding [130] investigated the asymptotic behavior of solutions to the linear Volterra integro-differential system

$$x_i'(t) = a_i(t) + b_i(t)x_i(t) + \sum_{j=1}^{n} \int_0^t K_{ij}(t,s)x_j(s)ds,$$

where $t \in \mathbb{R}_+$, $i = 1, 2, ..., n$. They proved that there exists a solution $x = (x_1, x_2, ..., x_n)^T : \mathbb{R}_+ \to \mathbb{R}^n$ for this system such that $x(t)$ is asymptotic to $(c_i + A_i(t))\beta_i(t)$, $t \to \infty, i = 1, 2, ..., n$, with $c = (c_1, c_2, ..., c_n)^T \in \mathbb{R}^n$ and $c_i + A_i(t) \geq 0$

provided that

$$0 < \liminf_{t \to \infty} \beta_i(t) \leq \limsup_{t \to \infty} \beta_i(t) < \infty, \ \ \liminf_{t \to \infty} (c_i + A_i(t)) > 0,$$

where $a_i, b_i : \mathbb{R}_+ \to \mathbb{R}$ and $K_{ij} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $i, j = 1, 2, ..., n$ are all continuous functions,

$$A_i(t) = \int_0^t \frac{a_i(s)}{\beta_i(s)} ds, \quad \beta_i(t) = \exp\left(\int_0^t b_i(s) ds\right), \ t \in \mathbb{R}_+,$$

$$A_i := \sup_{t \in \mathbb{R}_+} |A_i(t)| < \infty, \ \ \text{and } 0 \leq \sum_{j=1}^{n} \int_0^\infty \left(\int_0^t \left| K_{ij}(t,s) \frac{\beta_j(s)}{\beta_i(s)} \right| ds \right) dt < 1.$$

2.2.2 Asymptotic Behavior of Solutions for Fractional Differential Equations

Recently, few papers discussed the issue of asymptotic behavior for some types of fractional differential equations, see [93, 84, 85, 86, 131, 88, 90, 91].

In [93], Băleanu and Mustafa proved that the solution of the nonlinear fractional differential problem

$$\begin{cases} \left(D_{0^+}^{\alpha} (x - x_0)\right)(t) = f(t, x(t)), \ \ 0 < \alpha < 1, \ t > 0, \\ x(0^+) = x_0, \ x_0 \in \mathbb{R}, \end{cases} \tag{2.4}$$

is asymptotic to $o\left(t^{a\alpha}\right)$ as $t \to \infty$, $0 < 1 - a < \alpha$. They assumed that

$$|f(t,x)| \leq h(t)\, g\left(\frac{|x|}{(t+1)^{\alpha}}\right),$$

and

$$t^{(q_3/q_1)[1-q_1(1-\alpha)]}\left\{\int_0^t [h(s)]^{q_2}\,ds\right\}^{q_3/q_2} \leq M(t+1)^{\alpha},\ t \geq 0,$$

for some sufficiently large constant M, $q_1, q_2, q_3 > 1$, $a \in (0,1)$, $g : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing function and the function $h : [0,\infty) \to [0,\infty)$ is continuous with

$$t^{(q_3/q_1)[1-q_1(1-\alpha)]}\, \|h\|^{q_3}_{L^{q_2}(0,t)} = O(t^{\alpha}) \text{ when } t \to \infty.$$

In 2010, Băleanu *et al.* discussed in [86] the long-time behavior of solutions to some linear fractional differential equations of order greater than one. They proved that the problem

$$\begin{cases} \left(D^{\alpha}_{0^+}x\right)'(t) = h(t)x(t), \quad 0<\alpha<1,\ t>0, \\ \lim\limits_{t\to 0^+}\left(t^{1-\alpha}x(t)\right) = c_0,\ c_0 \in \mathbb{R}, \end{cases}$$

where $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order α, has a solution $x \in C((0,\infty),\mathbb{R})$ such that

$$\lim_{t\to\infty}\left(t^{1-\alpha}x(t)\right) = c_1, c_1 \in \mathbb{R},$$

and it has the asymptotic expansion

$$x(t) = (c_0 + O(1))\, t^{\alpha-1} + \left(\frac{c_1}{\Gamma(\alpha+1)} + o(1)\right) t^{\alpha} \text{ when } t \to \infty,$$

provided that the functional coefficient $h(t)$ satisfies the conditions: $h : (0, \infty) \to \mathbb{R}$ is continuous such that for some $T > 0$,

$$\int_T^{\infty} s^{\alpha+1} |h(s)| \, ds < \infty,$$

and

$$max\{1, T\} \int_{0^+}^{T} \frac{|h(s)|}{s^{1-\alpha}} ds + \int_T^{\infty} s^{\alpha} |h(s)| \, ds < \Gamma(\alpha + 1).$$

Also, the authors in [84] established under the condition

$$|f(t, x)| \leq \phi\left(t, \frac{|x|}{(1+t)^{\alpha}}\right), \ t \geq 0, \ x \in \mathbb{R},$$

($f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous function and it is assumed nondecreasing in the second argument), that the solution of the nonlinear fractional differential equation

$$\left(D_{0^+}^{\alpha} x'\right)(t) + f(t, x) = 0, \ 0 < \alpha < 1, \ t > 0,$$

can be expressed asymptotically as $c_1 + c_2 t^{\alpha} + O(t^{\alpha-1})$ when $t \to \infty$, $c_1, c_2 \in \mathbb{R}$.

In [90] and [91], Medved studied the problem of asymptotic integration of nonlinear higher order fractional differential equations of Caputo type.

In 2012, Medved [91] studied the behavior of solutions for the fractional differential problem

$$\begin{cases} \left({}^{C}D_{a^+}^{\alpha}x\right)(t) = f\left(t, x\left(t\right)\right), \quad t \geq a > 1,\ 1 < \alpha < 2, \\ x\left(a\right) = c_0,\ x'\left(a\right) = c_2. \end{cases} \tag{2.5}$$

He showed that problem (2.5) has a solution asymptotic to $b + ct$ as $t \to \infty$, for some b, $c \in \mathbb{R}$, provided that

1. $f(t, u)$ is continuous in $D = \{(t, u) : t \in [0, \infty),\ v \in \mathbb{R}\}$,

2. There are continuous nonnegative functions $h : \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ and g is nondecreasing with $\gamma > 0$ and $p(\gamma - 1) + 1 > 0$ such that

$$|f\left(t, u\right)| \leq t^{\gamma - 1} h\left(t\right) g\left(\frac{|u|}{t}\right),\ t > 0,\ \ (t, u) \in D,$$

where $p > 1$, $p\left(\alpha - 2\right) + 1 > 0$, $q = \frac{p}{p-1}$, $\gamma = 3 - \alpha + \frac{1}{p}$,

$$\int_a^{\infty} h^q\left(s\right) ds < \infty \ \text{ and } \int_a^{\infty} \frac{s^{q-1}}{g^q\left(s\right)} ds = \infty.$$

The fractional differential equation of Caputo type

$$\left({}^{C}D_{a^+}^{\alpha+1}x\right)(t) = f(t, x(t), x'(t)),\ a \geq 1\ ,\ 0 < \alpha < 1,$$

has been studied in [91] with analogous conditions to the above ones. It has been proved also that all solutions of this equation are asymptotic to $c_1 t + c_2$, $c_1, c_2 \in \mathbb{R}$.

In [90], Medved also showed that all solutions of the initial value problem

$$\begin{cases} \left({}^C D_{a^+}^{\alpha+1} x\right)(t) = f(t, x(t)),\ t \geq a > 1,\ 0 < \alpha < 1, \\ x(a) = c_1, x'(a) = c_2, \end{cases}$$

where ${}^C D_{a^+}^{\alpha+1}$ is the Caputo fractional derivative of order $\alpha + 1$, are asymptotic to a line for large values of t, provided that the following conditions hold:

1. $f(t, u)$ is continuous in $\Lambda = \{(t, u) : t \in [0, \infty), v \in \mathbb{R}\}$.

2. There are continuous nonnegative function $h : \mathbb{R}_+ \to \mathbb{R}_+$, continuous nonnegative nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$, and real numbers $\lambda > 0$, $p > 1$ $q = \frac{p}{p-1}$ with $p(\lambda - 1) + 1 > 0$, $\lambda = 2 - \alpha - \frac{1}{p}$ such that

$$|f(t, x)| \leq t^{\lambda-1} h(t) g\left(\frac{|x|}{t}\right), t > 0, \quad (t, x) \in \Lambda,$$

$$\int_a^\infty h^q(s) ds < \infty \ \text{ and } \int_a^\infty \frac{t^{q-1} dt}{g^q(t)} = \infty.$$

In 2015, Medved and Pospíšsil considered in the paper [92] a more general case when the right-hand side depends on Caputo fractional derivatives of the solution. They proved that there exists a constant $b \in \mathbb{R}$ such that any global solution of the initial value problem

$$\begin{cases} \left({}^C D_{a^+}^{\alpha} x\right)(t) = f\left(t, x\left(t\right), x'\left(t\right), ..., x^{(n-1)}\left(t\right), \left({}^C D_{a^+}^{\alpha_1} x\right)(t), ..., \left({}^C D_{a^+}^{\alpha_m} x\right)(t)\right), \\ x^{(i)}\left(a\right) = c_i,\ i = 0, 1, ..., n-1,\ n \in \mathbb{N}, \end{cases}$$

where $t \geq a$ and $n - 1 < \alpha_j < \alpha < n$, $j = 1, 2, ..., m, m \in \mathbb{N}$, is asymptotic to bt^r

with $r = \max\{n-1, \alpha_m\}$.

2.2.3 Asymptotic Behavior of Solutions for Fractional Integro-differential Equations

In 2004, Momani, *et al.* [132] discussed the Lyapunov stability and asymptotic stability conditions for the solutions of the fractional integro-differential equations

$$\left(D_{a^+}^{\alpha} x\right)(t) = f(t, x(t)) + \int_a^t k(t, s, x(s))ds,\ 0 < \alpha \leq 1,\ t \geq a, \qquad (\mathrm{E}_2)$$

with the initial condition $\left(I_{a^+}^{1-\alpha} x\right)(a^+) = c_0 \in \mathbb{R}$. The assumptions

$$|f(t, x(t))| \leq \gamma(t)\,|x|\,,$$

$$\int_s^t k(\sigma, s, x(s))d\sigma \leq \delta(t)\,|x|\,, \quad s \in [a, t],$$

where $\gamma(t)$ and $\delta(t)$ are continuous nonnegative functions and

$$\sup \int_a^t (t-s)^{\alpha-1}[\gamma(s) + \delta(s)]ds < \infty,$$

were imposed. The authors proved that every solution $x(t)$ of (E_2) satisfies

$$|x(t)| \leq \frac{|c_0|}{\Gamma(\alpha)}(t-a)^{\alpha-1} \exp\left\{\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}[\gamma(s) + \delta(s)]ds\right\} < \infty,$$

and if

$$\int_a^t (t-s)^{\alpha-1}[\gamma(s)+\delta(s)]ds = O\left((t-a)^{\alpha-1}\right),$$

then $|x(t)| \leq C_0(t-a)^{\alpha-1}$ where C_0 is a positive constant, and hence the solution of (E_2) is asymptotically stable.

Furati and Tatar [104] considered the equation (E_2) subject to the initial condition

$$\lim_{t\to a^+}\left(t^{1-\alpha}x(t)\right) = b,\ b\in\mathbb{R},\ 0<\alpha<1,\ a=0,$$

and showed that solutions decay polynomially for some nonlinear functions f and k. When $k\equiv 0$, they proved in [131] that solutions of the problem exist globally and decay as a power function in the space $C^{\alpha}_{1-\alpha}[0,\infty)$ defined in (4.65), see Section 4.5. In 2007, the same authors considered in [88] the equation (E_2) and found bounds for solutions on infinite time intervals and also provided sufficient conditions assuring decay of power type for the solutions.

In 2014, Brestovanska and Medved studied in [87] the initial-value problem

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) + \sum\limits_{i=1}^{m} h_i(t)\int_0^t (t-s)^{\alpha_i-1} g_i(\tau, x(\tau), x'(\tau))\, d\tau = 0, \\ x(1) = c_1, \quad x'(1) = c_2, \quad c_1, c_2 \in \mathbb{R}, \end{cases}$$

where $0<\alpha_i<1$, $i=1,2,...,m$, $m\in\mathbb{N}$ and $t>0$. Sufficient conditions for all solutions of the above problem to be asymptotic at infinity to a line are found.

2.3 Nonexistence Results

2.3.1 Nonexistence Results for Differential and Integrodifferential Equations of Integer Order

There are many results in the literature regarding the nonexistence of solutions and blow up of solutions for differential, integral and integro-differential equations of integer orders (ordinary or partial), see e.g. [133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147].

For the nonlinear integral equations especially Volterra integral equations, we refer the reader to [148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164] and the references therein. A comprehensive bibliography can be found in [165]. Some recent results on blow-up solutions for nonlinear Volterra integral equations of the second kind are surveyed in [159].

It is well-known that the Bernoulli differential equation

$$x'(t) + x(t) = x^q(t), \ t > 0, \ q > 1, \tag{2.6}$$

has the solution

$$x(t) = \left(\left(x_0^{1-q} - 1\right) e^{(q-1)t} + 1\right)^{\frac{1}{1-q}}, \ x_0 = x(0),$$

which blows up in finite time

$$T_b = \frac{1}{1-q} \ln \left(1 - x_0^{1-q}\right)$$

if and only if the initial data $x_0 > 1$, (see e.g. [165]).

The nonlinear second-order ordinary differential equation

$$x^{''}(t) = x^q(t), \tag{2.7}$$

subject to

$$x(0) = x_0,\ x'(0) = x_1,$$

has the solution

$$x\,(t) = \left(\frac{1-q}{2}\sqrt{\frac{2}{q+1}}t + x_0^{\frac{1-q}{2}}\right)^{\frac{2}{1-q}}, \tag{2.8}$$

when $x_1 = \sqrt{\frac{2}{q+1}x_0^{q+1}}$, $x_0 > 0$, and this solution blows up in finite time

$$T_b = \frac{2}{q-1}\sqrt{\frac{q+1}{2}}x_0^{\frac{1-q}{2}}. \tag{2.9}$$

The nonlinear Volterra integro-differential equation

$$x'(t) = -c + \int_0^t x^q(s)ds,\ t > 0,\ q > 1, \tag{2.10}$$

can be transformed by differentiation into (2.7).

When $c = \sqrt{\frac{2}{q+1}x_0^{q+1}}$, $x_0 > 0$, the solution of (2.10) is given by (2.8)and it blows up in the finite time (2.9).

Agarwal *et al.*, showed in [133] that the integro-differential inequalities

$$(-1)^{2^n} x^{(2^n)}(t) \geq \int_0^t k(t-s, x(s))\, ds, \quad t \geq 0, \; n = 0, 1, 2, ...,$$

have no positive solutions on $[0, \infty)$ for $n = 0$ and no bounded positive solutions on $[0, \infty)$ for $n \geq 1$ if the inequalities

$$-\mu^{2^n} + \int_0^\infty e^{\mu s} h(s)\, ds > 0, \text{ for all } \mu > 0, \; n = 0, 1, 2, ...,$$

hold, where $k : C[0, \infty) \times (0, \infty) \to [0, \infty)$ and $h : C[0, \infty) \to [0, \infty)$ satisfy the following:

1. $k : C[0, \infty) \times (0, \infty) \to [0, \infty)$ such that $k(t, u) \neq 0$ on $J \times (0, \infty)$ for some subinterval J of $[0, T]$, $T > 0$.

2. $h : C[0, \infty) \to [0, \infty)$ is not identically zero on $[0, T]$.

3. There exists a positive constant T_1 such that $\frac{k(t,u)}{u} \geq h(t)$ for $(t, u) \in [0, T) \times (0, T_1)$.

In [151], Małolepszy and Okrasiński, considered the nonlinear Volterra integral equation

$$x(t) = \int_0^t k(t-s) f(x(s))\, ds, \; t \geq 0, \tag{2.11}$$

where the function $f : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing and $f(0) = 0$. The kernel $k : (0, \infty) \to [0, \infty)$ is positive and locally integrable such that $\lim_{t\to\infty} K(t) := \int_0^t k(s)ds = \infty$. They proved that solutions of (2.11) blow up if g is a strictly increasing, positive and continuous function with $g(t) < f(t)$ for $t \in (c,\infty)$, $g(c) = f(c)$, $\lim_{t\to\infty} \frac{t}{g(t)} = 0$ and the series

$$\sum_{m=0}^{\infty} K^{-1}\left(\frac{\left(g^{-1}\circ f\right)^m(t)}{g\left(\left(g^{-1}\circ f\right)^m(t)\right)}\right)$$

converges for some point $t \in (c,\infty)$. The solution x of (2.11), they considered, is assumed to be continuous with $x(0) = 0$ and $x > 0$ in $(0,T)$.

When the function

$$f(t) = \begin{cases} f_1(t), & 0 \le t < a, \\ pt^q, & t \ge a, \end{cases} \quad , \ p > 0,\ q > 1,$$

is positive, absolutely continuous and strictly increasing such that $f(0) = 0$ and $\lim_{t\to 0^+} \frac{t}{f(t)} = 0$, the same authors found in [150] that the convergence of the integral

$$\int_0^t \frac{K^{-1}(\tau)}{\tau \ln \tau} d\tau, \tag{2.12}$$

for any $t \in (0,1)$ is a necessary and sufficient condition for existence of blow-up solutions to equation (2.11).

Based on the above results and other related results from [156, 157], the authors also established in 2010 lower and upper estimates of the blow-up time for solutions of (2.11), (see [149]).

In 2012, Małolepszy and Niedziela, showed that every nontrivial solution of equation (2.11) with f given by

$$f(t) = \begin{cases} t^{q_1}, & 0 \le t < 1,\ 0 < q_1 < 1, \\ t^{q_2}, & t \ge 1,\ q_2 > 1, \end{cases}$$

blows up in finite time. They relaxed the condition $\lim_{t\to\infty} K(t) = \infty$ to $\lim_{t\to\infty} K(t) > 1$ and the integral (2.12) to converge only for a sufficiently small t. By a nontrivial solution, they mean a continuous function x defined on $[0, T)$ such that $x(0) = 0$ and $x(t) > 0$ for all $t \in (0, T)$, where $[0, T)$ is the maximal interval of existence and $0 < T \le \infty$.

Mydlarczyk proved in [152] that the positive continuous solution of

$$x(t) = \int_0^t (t-s)^{\alpha-1} f_1(s)\, f_2(x(s) + f_3(s))\, ds, \quad t, \alpha > 0,$$

blows up if and only if

$$\int_\nu^\infty \left(\frac{s}{f_2(s)}\right)^{\frac{1}{\alpha}} \frac{ds}{s} < \infty, \text{ for any } \nu > 0,$$

where $f_i, (i = 1, 2, 3)$ are nondecreasing and continuous for $t \neq 0$, $f_i(t) = 0$ for $t \le 0$ and $f_i(t) > 0$ for $t > 0$. For $f_3 \equiv 0$, he gave the following implicit form for lower and upper bounds on the blow-up time T_b,

$$\alpha \int_0^{T_b} f_1^{\frac{1}{\alpha}}(s)\, ds \le \int_0^\infty \left(\frac{s}{f_2(s)}\right)^{\frac{1}{\alpha}} \frac{ds}{s} \le (\alpha+1)\, T_b f_1^{\frac{1}{\alpha}}(T_b), 0 < \alpha \le 1,$$

$$\frac{\alpha\left(2^{\frac{1}{\alpha}}-1\right)}{2^{\frac{1}{\alpha}}}\int_0^{\frac{T_b}{2}} f_1^{\frac{1}{\alpha}}(s)\,ds \le \int_0^{\infty}\left(\frac{s}{f_2(s)}\right)^{\frac{1}{\alpha}}\frac{ds}{s} \le \alpha\int_0^{T_b} f_1^{\frac{1}{\alpha}}(s)\,ds,\ \ \alpha>1.$$

He also studied the asymptotic growth of the blowing up solution near the blow-up time T_b and showed that there exist constants $a_1, a_2 > 0$ such that

$$\frac{1}{3}F^{-1}\left(a_1\left(T_b-t\right)\right) \le x(t) \le F^{-1}\left(a_2\left(T_b-t\right)\right),\ \ t\to T_b^-,$$

where F^{-1} is the inverse function of

$$F(\nu)=\int_{\nu}^{\infty}\left(\frac{s}{f_2(s)}\right)^{\frac{1}{\alpha}}\frac{ds}{s}.$$

In 2011, Ma showed in [141] that the solution of the nonlinear VIDE

$$x'(t)+x(t)=\int_0^t k(t-s)f\left(x(s)\right)ds,\quad t>0, \tag{2.13}$$

blows up in finite time if and only if for some $\beta > 0$,

$$\int_{\nu}^{\infty}\left(\frac{s}{f(s)}\right)^{\frac{1}{\beta}}\frac{ds}{s}<\infty,\ \text{ for any } \nu>0, \tag{2.14}$$

where $k(t)$ is a positive and locally integrable function with $\lim\limits_{t\to\infty}\int_0^t k(s)ds=\infty$. The function $f(t)$ is assumed to be continuous, nonnegative and nondecreasing for $t>0$, $f\equiv 0$ for $t\le 0$, and $\lim\limits_{t\to\infty}\frac{f(t)}{t}=\infty$. Clearly, if $f(x(s))=|x(s)|^q$ in the equation (2.13), the condition (2.14) simply means that $q>1$.

Brunner and Yang (2012) considered in [148] the blow-up behavior of the

Hammerstein-Volterra integral equation

$$x(t) = g\left(t\right) + \int_0^t k(t-s) f\left(s, x\left(s\right)\right) ds,\ \ t > 0, \tag{2.15}$$

where

(i) $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function with $f(t, 0) \equiv 0$ and

$$f(t_1, x_1) < f(t_2, x_2) \text{ for } \ (0, 0) < (t_1, x_1) \leq (t_2, x_2) \text{ and } x_1 \neq x_2,$$

(ii) $\lim\limits_{u\to\infty} \frac{f(0,u)}{u} = \infty$,

(iii) $g(t)$ is a positive, nondecreasing, continuous function,

(iv) the kernel $k\left(t\right) = t^{\beta-1} h\left(t\right), \beta > 0$, $h\left(t\right) \geq 0$ is bounded in any finite interval and $\inf\limits_{t\in[0,\sigma]} h\left(t\right) > 0$ for some $\sigma > 0$.

They proved, under the conditions (i)-(iv), that the solution of (2.15) blows up in finite time if and only if there exists t_1 such that

$$g\left(t_1\right) + \min_{u\in[0,\infty)} \left[\int_0^{t_1} k(t_1 - s) f\left(s, u\right) ds - u\right] > 0,$$

$$\int_\nu^\infty \left(\frac{s}{f\left(t_1, s\right)}\right)^{\frac{1}{\beta}} \frac{ds}{s} < \infty \text{ for all } \nu > 0.$$

In 2013, the same authors generalized this result to the delayed Hammerstein-

Volterra integral equation (see [164])

$$x(t) = g\left(t\right) + \int_{\phi(t)}^{t} k(t-s) f\left(s, x\left(s\right)\right) ds,\ t \geq t_0,$$

where the delay function $\phi\left(t\right)$ is continuous with $t_0 - \tau \leq \phi\left(t\right) < t$ for all $t \in (t_0, \infty)$ and $\tau \geq 0$. Also, $x(t) = \psi\left(t\right)$, $t_0 - \tau \leq t < t_0$ is an initial given function.

2.3.2 Nonexistence Results for Fractional Differential and Integro-differential Equations

For the fractional integro-differential case we cannot find much. The few works we are aware of are concerned with a polynomial source. This is of course a special case corresponding to the Dirac delta function in our case.

In the last two decades the nonexistence of local and global solutions and blowing-up solutions are investigated for ordinary and partial fractional differential equations, we refer to [166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183] and the references therein.

In 2010, Laskri and Tatar investigated nonexistence of global solutions for the Cauchy problem

$$\begin{cases} \left(D_{0^+}^{\alpha} x\right)(t) \geq t^{\beta} \left|x(t)\right|^{q},\ \ t > 0,\ 0 < \alpha < 1,\ q > 1, \\ \left(D_{0^+}^{\alpha-1} x\right)(0^+) = x_0,\ x_0 \in \mathbb{R}, \end{cases} \tag{2.16}$$

in the space

$$\mathbf{L}_\alpha\left(0,b\right)=\left\{x\in L^1\left(0,b\right),\ D^\alpha_{0+}x\in L^1\left(0,b\right)\right\},\quad \alpha,b>0.$$

They found that problem (2.16) does not admit nontrivial global $\mathbf{L}_\alpha$ solutions when $\beta>-\alpha$, $1<q\leq\frac{\beta+1}{1-\alpha}$ and $x_0\geq 0$. They also showed that the exponent $\frac{\beta+1}{1-\alpha}$ is critical (see [180]).

In 2013, Furati, Kassim and Tatar considered in [168] the problem

$$\begin{cases}\left(D^{\alpha,\beta}_{0^+}x\right)(t)\geq t^\gamma\left|x(t)\right|^q, & t>0,\ q>1,\ \gamma\in\mathbb{R},\\ \left(D^{\lambda-1}_{0^+}x\right)(0^+)=x_0>0,\ \lambda=\alpha+\beta-\alpha\beta, & \end{cases}\tag{2.17}$$

where

$$D^{\alpha,\beta}_{0^+}:=I^{\beta(1-\alpha)}_{0^+}\frac{d}{dt}I^{1-\lambda}_{0^+},\ 0<\alpha<1,\ 0\leq\beta\leq 1,$$

and I^ρ_{0+} is the Riemann-Liouville fractional integral of order $\rho>0$. Using the test function method introduced in [184], they proved that when $\gamma>-\alpha$, $1<q<\frac{1+\gamma}{1-\alpha}$ and $x_0>0$, the problem (2.17) has no global nontrivial solution in the weighted space of continuous functions $C^\lambda_{1-\lambda}\left[0,b\right]$. Furati and Kirane showed in [167] that the fractional differential equation

$$x'(t)+\left({}^C D^\alpha_{0+}x\right)(t)=\left|x(t)\right|^q,\quad t>0,\ 0<\alpha<1,\ q>1,\tag{2.18}$$

admits no global solutions when $x(0)=x_0>0$.

In 2014, Kadem *et al.*, proved in [169] that the global solutions of the following fractional differential systems with exponential nonlinearities

$$\begin{cases} x'(t) + p\left({}^{C}D_{0^+}^{\alpha}x\right)(t) = e^{y(t)}, & t > 0, \quad p > 0,\ 0 < \alpha < 1, \\ y'(t) + q\left({}^{C}D_{0^+}^{\beta}y\right)(t) = e^{x(t)}, & t > 0, \quad q > 0,\ 0 < \beta < 1, \end{cases}$$

$$\begin{cases} p\left({}^{C}D_{0^+}^{\alpha}x\right)(t) = \lambda e^{y(t)}, \quad t > 0, \quad p > 0,\ 0 < \alpha < 1, \\ q\left({}^{C}D_{0^+}^{\beta}y\right)(t) = \lambda e^{x(t)}, \quad t \geq 0, \quad q > 0,\ \frac{1}{2} \leq \lambda \leq 1,\ 0 < \beta < 1, \end{cases}$$

subject to the initial conditions $x(0) = x_0 > 0, y(0) = y_0 > 0$, blow up in a finite time. They also provided bounds on the blow-up time of the solution for each system. Using the methods developed in [162], they discussed the asymptotic behavior of the solutions of the systems near the blow-up by working on their corresponding systems of Volterra integral equations.

CHAPTER 3

PRELIMINARIES

In this chapter we briefly introduce some basic definitions, notions and properties from the theory of fractional calculus and fractional differential equations. We present some preliminary results related to the fractional integrals and derivatives to be used in the next chapters. More details about the theory of fractional integrals and derivatives can be found in [3, 4, 2].

3.1 Some Function Spaces

This section contains definitions of some important spaces such as the spaces of p-integrable, continuous, continuously differentiable, absolutely continuous and some weighted spaces of continuous functions. We present also some relations and properties of these spaces that will be used later in formulating our theorems and proofs.

Definition 3.1 *[3] Let* $-\infty \leq a < b \leq \infty$. *The space* $L^p(a,b)$ $(1 \leq p \leq \infty)$

consists of all Lebesgue real-valued measurable functions f *on* (a,b) *for which* $\|f\|_p < \infty$, *where*

$$\|f\|_p = \left(\int_a^b |f(s)|^p \, ds\right)^{1/p}, \quad 1 \le p < \infty,$$

$$\|f\|_\infty = ess \sup_{a \le t \le b} |f(t)|,$$

and $ess \sup |f(t)|$ *is the essential supremum of the function* $|f(t)|$.

Definition 3.2 *[185] The space* $L^1_{loc}(a,b)$ *consists of all Lebesgue measurable functions* $f : (a,b) \to \mathbb{R}$ *for which* $\|f\|_1 < \infty$ *on all strictly contained subsets* Ω *of* (a,b).

Definition 3.3 *[3] We denote by* $C[a,b]$ *and* $C^n[a,b]$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, *the spaces of continuous and* $n-$*times continuously differentiable functions on* $[a,b]$,*with the norms*

$$\|f\|_C = \max_{t \in [a,b]} |f(t)|,$$

$$\|f\|_{C^n} = \sum_{i=0}^{n} \left\|f^{(i)}\right\|_C = \sum_{i=0}^{n} \max_{t \in [a,b]} \left|f^{(i)}(t)\right|, \quad n \in \mathbb{N}_0,$$

respectively, where $C[a,b] = C^0[a,b]$.

Definition 3.4 *[186] A function* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous if for every positive number* ϵ, *there is a positive number* δ *such that whenever a finite sequence of pairwise disjoint sub-intervals* (a_i, b_i) *of* $[a,b]$ *satisfies* $\sum_i (b_i - a_i) < \delta$ *then* $\sum_i (f(b_i) - f(a_i)) < \epsilon$. *We denote by* $AC[a,b]$ *the space of absolutely continuous functions on* $[a,b]$.

It is known (see [3]) that $AC\left[a,b\right]$ coincides with the space of primitives of Lebesgue integrable functions

$$f\in AC[a,b]\Leftrightarrow f\left(t\right)=c+\int_{a}^{t}g\left(s\right)ds,\quad g\in L^{1}\left(a,b\right).$$

Therefore the absolutely continuous function f has an integrable derivative $f'=g$ almost everywhere on $\left[a,b\right]$ and $c=f\left(a\right)$. In general, we have the following definition.

Definition 3.5 *For $n\in\mathbb{N}$, we denote by $AC^{n}\left[a,b\right]$ the space of all real-valued functions which have continuous derivatives up to order $n-1$ on $\left[a,b\right]$ such that $D^{n-1}f\in AC\left[a,b\right]$, $D=\frac{d}{dt}$, that is*

$$AC^{n}\left[a,b\right]=\left\{f:\left[a,b\right]\rightarrow\mathbb{R}\text{ such that }D^{n-1}f\in AC\left[a,b\right]\right\}.$$

Note that $AC\left[a,b\right]\subset C\left[a,b\right]$ and $AC^{n+1}\left[a,b\right]\subset C^{n}\left[a,b\right]\subset AC^{n}\left[a,b\right],\ n\geq1$.

Definition 3.6 *[3] We denote by $C_{\gamma}\left[a,b\right],\ 0\leq\gamma<1$, the following weighted space of continuous functions*

$$C_{\gamma}\left[a,b\right]=\left\{f:\left(a,b\right]\rightarrow\mathbb{R}:\left(t-a\right)^{\gamma}f(t)\in C\left[a,b\right]\right\},\tag{3.1}$$

with the norm

$$\left\|f\right\|_{C_{\gamma}}=\left\|\left(t-a\right)^{\gamma}f\left(t\right)\right\|_{C},$$

In particular, $C\left[a,b\right]=C_{0}\left[a,b\right]$.

Definition 3.7 *[3] For $n \in \mathbb{N}$ and $0 \leq \gamma < 1$, we denote by $C_\gamma^n[a,b]$, the following weighted space of continuously differentiable functions up to order $n-1$ with nth derivative in $C_\gamma[a,b]$,*

$$C_\gamma^n[a,b] = \left\{ f : (a,b] \to \mathbb{R} \mid f \in C^{n-1}[a,b],\ f^{(n)} \in C_\gamma[a,b] \right\},$$

with the norm

$$\|f\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \left\| f^{(k)} \right\|_C + \left\| f^{(n)} \right\|_{C_\gamma}.$$

In particular, $C_\gamma[a,b] = C_\gamma^0[a,b]$.

We have the following characterization of the space $C_\gamma^n[a,b]$.

Lemma 3.1 *[3] Let $n \in \mathbb{N}$ and $0 \leq \gamma < 1$. The space $C_\gamma^n[a,b]$ consists of those and only those functions f which can be represented in the form*

$$f(t) = (I_a^n g)(t) + \sum_{k=0}^{n-1} c_k (t-a)^k,$$

where $g \in C_\gamma[a,b]$,

$$(I_a^n g)(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} g(s)\, ds,$$

and c_k $(k = 0, 1, ..., n-1)$ are arbitrary constants.

Moreover,

$$g = f^{(n)}, \qquad c_k = \frac{f^{(k)}(a)}{k!} \qquad (k = 0, 1, ..., n-1).$$

Note that $C_\gamma^n[a,b] \subset AC^n[a,b]$, $n \geq 1$. Furthermore, we have the following relationship.

Lemma 3.2 *[187] Let $0 \leq \gamma < 1$, $n \geq 1$. Then, $f^{(n)} \in C_\gamma[a,b]$ if and only if $f \in C_\gamma^n[a,b]$.*

From the definitions of the spaces $C^n[a,b]$, $C_\gamma[a,b]$ and $C_\gamma^n[a,b]$, we have the following continuous embedding.

Lemma 3.3 *[3] The following embedding holds:*

$$C^n[a,b] \to C_{\mu_1}^n[a,b] \to C_{\mu_2}^n[a,b], \; n \in \mathbb{N}_0, \; 0 \leq \mu_1 \leq \mu_2 < 1,$$

with

$$\|f\|_{C_{\mu_2}^n} \leq K \|f\|_{C_{\mu_1}^n}, \quad K = \min\left[1, (b-a)^{\mu_2-\mu_1}\right].$$

In particular,

$$C[a,b] \to C_{\mu_1}[a,b] \to C_{\mu_2}[a,b],$$

with

$$\|f\|_{C_{\mu_2}} \leq (b-a)^{\mu_2-\mu_1} \|f\|_{C_{\mu_1}}.$$

3.2 Fractional Integrals and Derivatives

In this section we introduce some notation, definitions and preliminary results from fractional calculus related to two derivatives we are going to use throughout this study, namely the Riemann-Liouville and Caputo fractional derivatives.

3.2.1 Riemann-Liouville Fractional Derivative

Definition 3.8 *The Riemann-Liouville left-sided and right-sided fractional integrals of order $\alpha > 0$ are defined by*

$$\left(I_{a+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \; a < t < b, \tag{3.2}$$

$$\left(I_{b-}^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds, \; a < t < b, \tag{3.3}$$

respectively, provided the right-hand sides exist. We define

$$I_{a+}^{0} u = I_{b-}^{0} u = u.$$

The function $\Gamma(\alpha)$ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \; \alpha > 0.$$

Definition 3.9 *The Riemann-Liouville left-sided and right-sided fractional derivatives of order $\alpha \geq 0$, are defined by*

$$\left(D_{a+}^{\alpha} u\right)(t) = D^n \left(I_{a^+}^{n-\alpha} u\right)(t), \; t > a, \tag{3.4}$$

$$\left(D_{b-}^{\alpha} u\right)(t) = (-1)^n D^n \left(I_{b^-}^{n-\alpha} u\right)(t), \; t < b, \tag{3.5}$$

respectively, where $D^n = \frac{d^n}{dt^n}$, $n = [\alpha]+1$, $[\alpha]$ is the integral part of α. In particular, *when* $\alpha = m \in \mathbb{N}_0$, *it follows from the definition that*

$$D_{a^+}^m u = D^m u,\ D_{b^-}^m u = (-1)^m D^m u.$$

Lemma 3.4 *[3] Let* $\alpha \geq 0$, $n = [\alpha] + 1$ *and* $0 \leq \gamma < 1$. *If* $u \in C_\gamma^n [a, b]$, *then the fractional derivatives* $D_{a^+}^\alpha u$ *and* $D_{b^-}^\alpha u$ *exist on* $(a, b]$ *and* $[a, b)$, *respectively.*

The next lemma shows that the Riemann-Liouville fractional integral and derivative of the power functions yield power functions multiplied by certain coefficients and with the order of the fractional derivative added or subtracted from the power.

Lemma 3.5 *[3] If* $\alpha \geq 0$, $\beta > 0$, *then*

$$\left(I_{a^+}^\alpha (s-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}, t > a,$$

$$\left(I_{b^-}^\alpha (b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-t)^{\beta+\alpha-1},\ t < b,$$

$$\left(D_{a^+}^\alpha (s-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1},\ t > a,$$

$$\left(D_{b^-}^\alpha (b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1}, t < b.$$

In particular, if $\beta = 1$, *then*

$$\left(D_{a+}^\alpha 1\right)(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)},\ \left(D_{b^-}^\alpha 1\right)(t) = \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Lemma 3.6 *[3] For $j = 1, 2, ..., n,\ n = [\alpha] + 1,$ we have*

$$\begin{aligned} \left(D_{a+}^{\alpha} (s-a)^{\alpha-j}\right)(t) &= 0,\ t > a, \\ \left(D_{b-}^{\alpha} (b-s)^{\alpha-j}\right)(t) &= 0,\ t < b. \end{aligned}$$

3.2.2 Caputo Fractional Derivative

The next fractional derivative we define is called the Caputo fractional derivative introduced by Caputo in his paper [188] in 1967 and adopted in the framework of the theory of linear viscoelasticity by Caputo and Mainardi [189]. Actually, this is the kind of fractional derivative which is commonly used by engineers. We present here its definition and some of its properties.

Definition 3.10 *The Caputo left-sided and right-sided fractional derivatives of order $\alpha \geq 0$, are defined by*

$$\begin{aligned} \left({}^{C}D_{a+}^{\alpha} u\right)(t) &= \left(D_{a+}^{\alpha}\left(u(s) - \sum_{j=0}^{n-1} \frac{u^{(j)}(a)}{j!}(s-a)^{j}\right)\right)(t), \\ \left({}^{C}D_{b-}^{\alpha} u\right)(t) &= \left(D_{b-}^{\alpha}\left(u(s) - \sum_{j=0}^{n-1} \frac{u^{(j)}(b)}{j!}(b-s)^{j}\right)\right)(t), \end{aligned}$$

respectively, where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ for $\alpha \in \mathbb{N}_0$. In particular, *when* $\alpha = n \in \mathbb{N}_0$, *it follows from these definitions that*

$$ {}^{C}D^{0}_{a^+}u = {}^{C}D^{0}_{b^-}u = u,\ \ {}^{C}D^{n}_{a^+}u = D^{n}u,\ \ {}^{C}D^{n}_{b^-}u = (-1)^{n}D^{n}u. $$

Note that if $u^{(j)}(a) = 0$ for all $j = 0, 1, ..., n-1$, then

$$ {}^{C}D^{\alpha}_{a^+}u = D^{\alpha}_{a^+}u, $$

and if $u^{(j)}(b) = 0$ for all $j = 0, 1, ..., n-1$, then

$$ {}^{C}D^{\alpha}_{b^-}u = D^{\alpha}_{b^-}u. $$

Lemma 3.7 *Let* $\alpha \geq 0$, $\beta > 0$, $n = [\alpha] + 1$ *for* $\alpha \notin \mathbb{N}_0$ *and* $n = \alpha$ *for* $\alpha \in \mathbb{N}_0$. *If* $u \in AC^{n}[a, b]$, *then* ${}^{C}D^{\alpha}_{a^+}u$ *and* ${}^{C}D^{\alpha}_{b^-}u$ *exist almost everywhere on* $[a, b]$ *and are represented by*

$$ \left({}^{C}D^{\alpha}_{a^+}u\right)(t) = \left(I^{n-\alpha}_{a^+}D^{n}u\right)(t), \tag{3.6} $$

$$ \left({}^{C}D^{\alpha}_{b^-}u\right)(t) = (-1)^{n}\left(I^{n-\alpha}_{b^-}D^{n}u\right)(t), \tag{3.7} $$

Lemma 3.8 *[3] Let* $\alpha > 0$, $\beta > 0$, $n = [\alpha] + 1$ *for* $\alpha \notin \mathbb{N}_0$ *and* $n = \alpha$ *for* $\alpha \in \mathbb{N}_0$. *Then, the following relations hold*

$$ \left({}^{C}D^{\alpha}_{a^+}\left(s-a\right)^{\beta-1}\right)(t) = \frac{\Gamma\left(\beta\right)}{\Gamma(\beta-\alpha)}\left(t-a\right)^{\beta-\alpha-1},\ \ \beta > n, $$

$$\left({}^{C}D_{b^-}^{\alpha}\,(b-s)^{\beta-1}\right)(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,(b-t)^{\beta-\alpha-1}\,,\ \ \beta>n,$$

$$\left({}^{C}D_{a^+}^{\alpha}\,(s-a)^{m}\right)(t)=0,\ \ \left({}^{C}D_{b^-}^{\alpha}\,(b-s)^{m}\right)(t)=0,\ \ m=0,1,...,n-1.$$

In particular, if $m=0$, then the Caputo fractional derivative of a constant is equal to zero.

Though the Riemann–Liouville fractional derivative is the most studied derivative in the fractional calculus, the Caputo fractional derivative is the most popular among physicists and engineers. The reason is that the differential equations modeled in terms of Caputo derivative require ordinary initial conditions just as ordinary differential equations do. On the contrary, fractional differential equations with Riemann–Liouville derivative require unusual initial conditions involving fractional integrals that are difficult to evaluate. This limits its applications in physics and other sciences but makes it more interesting and challenging for mathematicians.

For more details about fractional integrals and fractional derivatives, the reader is referred to the books [15, 44, 2, 3].

3.3 Preliminary Results

In this section we present some other important definitions, lemmas, theorems and properties which justify the assumption, tools and the methods utilized in our results later.

3.3.1 Fractional Calculus Lemmas

Lemma 3.9 *[2] (Fractional Integration by Parts) Let* $\alpha \geq 0$, $m_1 \geq 1, m_2 \geq 1$ *and* $\frac{1}{m_1} + \frac{1}{m_2} \leq 1 + \alpha$ *(*$m_1 \neq 1$ *and* $m_2 \neq 1$ *in the case when* $\frac{1}{m_1} + \frac{1}{m_2} = 1 + \alpha$*). If* $\varphi_1 \in L^{m_1}(a, b)$ *and* $\varphi_2 \in L^{m_2}(a, b)$*, then*

$$\int_a^b \varphi_1(t)\left(I_{a+}^{\alpha}\varphi_2\right)(t)\,dt = \int_a^b \varphi_2(t)\left(I_{b-}^{\alpha}\varphi_1\right)(t)\,dt.$$

The Riemann fractional integration operator I_{a+}^{α} has the semigroup property expressed in the following lemma.

Lemma 3.10 *[3] Let* $\alpha > 0$, $\beta > 0$ *and* $0 \leq \gamma < 1$*. Then*

$$I_{a+}^{\alpha} I_{a+}^{\beta} u = I_{a+}^{\alpha+\beta} u,$$

almost everywhere in $[a, b]$ *for* $u \in L^p(a, b)$ *and holds at any point in* $(a, b]$ *if* $u \in C_\gamma[a, b]$*. When* $u \in C[a, b]$*, then this relation is valid at every point in* $[a, b]$*.*

The fractional differentiation is an inverse operation of the fractional integration from the left as shown in the following lemma.

Lemma 3.11 *[3] Let* $\alpha > 0$ *and* $0 \leq \gamma < 1$*. If* $u \in C_\gamma[a, b]$*, then the relation*

$$D_{a+}^{\alpha} I_{a+}^{\alpha} u = u$$

holds at every point in $(a, b]$*. When* $u \in C[a, b]$ *this relation is valid at every point* *in* $[a,b]$.

Lemma 3.12 *[3] Let* $0 < \beta < \alpha$ *and* $0 \leq \gamma < 1$. *If* $u \in C_\gamma\ [a,b]$, *then*

$$D^{\beta}_{a^+} I^{\alpha}_{a^+} u = I^{\alpha-\beta}_{a^+} u$$

at every point in $(a,b]$. *When* $u \in C\ [a,b]$, *this relation is valid at every point in* $[a,b]$. *In particular, if* $\beta = m \in \mathbb{N}$ *and* $\alpha > m$, *then* $D^m I^{\alpha}_{a^+} u = I^{\alpha-m}_{a^+} u$.

Lemma 3.13 *[98] Let* $n-1 < \alpha < n$, $m-1 < \beta < m$ *and* $\beta < \alpha$.

1. *If* $u \in C\,[a,b]$, *then*

$$^{C}D^{\beta}_{a+} I^{\alpha}_{a^+} u = I^{\alpha-\beta}_{a^+} u$$

   at every point in $[a,b]$,

2. *If* $u \in C^{\ n-1}\,[a,b]$ *and* $^{C}D^{\alpha}_{a+} u \in C\,[a,b]$, *then* $^{C}D^{\beta}_{a+} u \in C\,[a,b]$.

Now we consider some other properties of the Riemann-Liouville fractional integral I^{α}_{a+} in the space $C_\gamma\,[a,b]$ defined in Definition 3.6.

Lemma 3.14 *[3] Let* $\alpha > 0$ *and* $0 \leq \gamma < 1$.

(i) *Then,* I^{α}_{a+} *is bounded from* $C_\gamma\,[a,b]$ *into* $C_\gamma\,[a,b]$.

(ii) *If* $\gamma \leq \alpha$, *then* I^{α}_{a+} *is bounded from* $C_\gamma\,[a,b]$ *into* $C\,[a,b]$.

Lemma 3.15 *[190] Let $0 \leq \gamma < 1$ and $\alpha > \gamma$. If $u \in C_\gamma [a,b]$, then*

$$\left(I^{\alpha}_{a+}u\right)\left(a^{+}\right) = \lim_{t \to a^{+}} \left(I^{\alpha}_{a+}u\right)(t) = 0.$$

The following result is about the composition $I^{\alpha}_{a+}D^{\alpha}_{a^+}$ of the Riemann-Liouville fractional integration and differentiation operators.

Lemma 3.16 *[3] Let $\alpha > 0$, $0 \leq \gamma < 1$, $n = [\alpha] + 1$. If $u \in C_\gamma \ [a,b]$ and $I^{n-\alpha}_{a^+}u \in C^n_\gamma [a,b]$, then*

$$\left(I^{\alpha}_{a+}D^{\alpha}_{a+}u\right)(t) = u(t) - \sum_{i=1}^{n} \frac{\left(D^{n-i}I^{n-\alpha}_{a^+}u\right)(a)}{\Gamma(\alpha - i + 1)} (t-a)^{\alpha - i}$$

for all $t \in (a,b]$. In particular, if $0 < \alpha < 1$, $u \in C_\gamma \ [a,b]$ and $I^{1-\alpha}_{a^+}u \in C^1_\gamma [a,b]$ then

$$\left(I^{\alpha}_{a+}D^{\alpha}_{a+}u\right)(t) = u(t) - \frac{\left(I^{1-\alpha}_{a^+}u\right)(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}. \tag{3.8}$$

The next Lemma is an analog of Lemma 3.16 for the Caputo fractional derivative.

Lemma 3.17 *[3] Let $\alpha > 0$ and $n = -[-\alpha]$. If $u \in AC^n [a,b]$ or $u \in C^n [a,b]$, then*

$$\left(I^{\alpha}_{a^+}{}^{C}D^{\alpha}_{a+}u\right)(t) = u(t) - \sum_{i=1}^{n-1} \frac{u^{(i)}(a)}{i!} (t-a)^{i},$$

for all $t \in (a,b]$. In particular, if $0 < \alpha \leq 1$ and $u \in AC [a,b]$ or $u \in C^1 [a,b]$, *then*

$$\left(I_{a^+}^{\alpha}\, {}^{C}D_{a^+}^{\alpha} u\right)(t) = u(t) - u(a), \quad t \in (a, b]\,. \tag{3.9}$$

3.3.2 Some Useful Inequalities

We mention here some useful basic inequalities and some linear and nonlinear integral inequalities to be used in the next chapters.

Lemma 3.18 *[191] If $\lambda,\ \nu,\ \omega > 0$, then, for any $t > 0$, we have*

$$\int_0^t (t-s)^{\nu-1} s^{\lambda-1} e^{-\omega s} ds \le C t^{\nu-1},$$

where C is a positive constant independent of t. In fact,

$$C = \max\left\{1, 2^{1-\nu}\right\} \Gamma(\lambda) \left(1 + \lambda(\lambda+1)/\nu\right) \omega^{-\lambda}.$$

The inequalities in the next lemma are of Jensen's inequalities type.

Lemma 3.19 *[192] Let $a_i,\ i = 1, ..., m,\ m \in \mathbb{N}$, be nonnegative real numbers. Then,*

$$\left(\sum_{i=1}^{m} a_i\right)^q \le m^{q-1} \sum_{i=1}^{m} a_i^q \quad \textit{for } q \ge 1.$$

Moreover, if $a_i > 0$ for all $i = 1, ..., m$, then

$$\left(\sum_{i=1}^{m} a_i\right)^q \ge m^{q-1} \sum_{i=1}^{m} a_i^q \quad \textit{for } \ 0 \le q \le 1.$$

Lemma 3.20 *[193] (Hölder's Inequality) Suppose that* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p}+\frac{1}{q} = 1$. *If* $u \in L^p(a,b)$ *and* $v \in L^q(a,b)$ *then* $uv \in L^1(a,b)$ *and*

$$\|uv\|_1 \leq \|u\|_p \|v\|_q \ .$$

Lemma 3.21 *[194] (Bihari's Inequality) Let* u *and* λ *be continuous and nonnegative functions on* $[a,\infty)$ *and* ω *be continuous and nondecreasing functions on* $[0,\infty)$ *and positive on* $(0,\infty)$. *If*

$$u(t) \leq c + \int_a^t \lambda(s)\omega(u(s))ds, \ t \in [a,\infty),$$

where c *is a positive constant, then*

$$u(t) \leq W^{-1}\left[W(c) + \int_a^t \lambda(s)ds\right], t \in [a, b_1],$$

where

$$W(v) = \int_{v_0}^{v} \frac{d\tau}{\omega(\tau)}, v > 0, v_0 > 0,$$

W^{-1} *is the inverse function of* W *and* b_1 *is chosen so that*

$$W(c) + \int_a^t \lambda(s)ds \in Dom\left(W^{-1}\right), \ \text{for all } t \in [a, b_1],$$

Dom here denotes the domain of the function W^{-1}.

Let $S \subset \mathbb{R}$ be a set. For two functions $f, g : S \to \mathbb{R}/\{0\}$, we say that $f \propto g$ if g/f is nondecreasing on S.

In his paper [124], Pinto extended the Bihari's inequality to two nonlinearities as shown in the following Lemma.

Lemma 3.22 *([124], Lemma 1) Let $u, \lambda_i, i = 1, 2$ be continuous and nonnegative functions on $I = [a, b]$ and ω_i, $i = 1, 2$ be continuous and nondecreasing functions on $[0, \infty)$ and positive on $(0, \infty)$ such that $\omega_1 \propto \omega_2$ and let c be a positive constant. If*

$$u(t) \leq c + \int_a^t \lambda_1(s)\omega_1(u(s))ds + \int_a^t \lambda_2(s)\omega_2(u(s))ds, \ t \in [a, b],$$

then, for $t \in [a, b_1]$,

$$u(t) \leq W_2^{-1}\left(W_2(c_1) + \int_a^t \lambda_2(s)ds\right),$$

where

1. $W_1(v) = \int_c^v \frac{d\tau}{\omega_1(\tau)}, v > 0, W_2(v) = \int_{v_0}^v \frac{d\tau}{\omega_2(\tau)} v > 0, v_0 > 0, i = 1, 2$ and W_i^{-1} is the inverse function of W_i.

2. The constants c_0 and c_1 are given by $c_0 = c$ and $c_1 = W_1^{-1}\left(\int_a^{b_1} \lambda_1(s)ds\right)$.

3. b_1 is the largest number such that $b_1 \geq a$ and

$$\int_a^{b_1} \lambda_i(s)ds \leq \int_{c_{i-1}}^{\infty} \frac{d\tau}{\omega_i(\tau)}, \ i = 1, 2.$$

By induction on n, Pinto generalized Lemma 3.22 to a finite number of nonlinearities.

Lemma 3.23 *([124], Theorem 1) Let $u, \lambda_i, i = 1, ..., n$ be continuous and nonnegative functions on $I = [a, b]$ and the functions ω_i, $i = 1, ..., n$ be continuous nonnegative and nondecreasing on $[0, \infty)$ such that $\omega_1 \propto \omega_2 \propto ... \propto \omega_n$. Assume further that c is a a positive constant. If*

$$u(t) \leq c + \sum_{i=1}^{n} \int_a^t \lambda_i(s)\omega_i(u(s))ds,\ t \in [a, b],$$

then, for $t \in [a, b_1]$,

$$u(t) \leq W_n^{-1}\left(W_n(c_{n-1}) + \int_a^t \lambda_n(s)ds\right),$$

where

1. $W_i(v) = \int_{v_i}^{v} \frac{d\tau}{\omega_i(\tau)}, v > 0, v_i > 0, i = 1, ..., n$ *and* W_i^{-1} *is the inverse function of* W_i.

2. *The constants c_i are given by $c_0 = c$ and $c_i = W_i^{-1}\left(W_i(c_{i-1}) + \int_a^{b_1} \lambda_i(s)ds\right)$, $i = 1, ..., n-1$.*

3. *The number $b_1 \in [a, b]$ is the largest number such that*

$$\int_a^{b_1} \lambda_i(s)ds \leq \int_{c_{i-1}}^{\infty} \frac{d\tau}{\omega_i(\tau)}, i = 1, ..., n.$$

Lemma 3.24 *([124], Theorem 4) Let u, λ_i, ω_i, $i = 1, 2, 3$ and c be as in Lemma*

3.23. If

$$u(t) \leq c + \int_a^t \lambda_1(s)\omega_1(u(s))ds + \int_a^t \lambda_2(s)\omega_2\left(\int_a^s \lambda_3(\tau)\omega_3(u(\tau))d\tau\right) ds,$$

then, for $t \in [a, b_1]$,

$$u(t) \leq W_3^{-1}\left(W_3(c_2) + \int_a^t \lambda_3(s)ds\right),$$

where W_i, W_i^{-1}, $i = 1, 2, 3$ *and* c_0, c_1, c_2 *are the same as in Lemma 3.23.*

CHAPTER 4

ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

This chapter is devoted to the study of the asymptotic behavior of solutions for the fractional integro-differential equation (1.1) with the Caputo and Riemann-Liouville fractional derivatives of orders $1 < \mu < 2$, $0 \leq \beta < \mu$ and $0 \leq \gamma < \mu$. We begin this chapter by discussing the existence and uniqueness of solutions of (1.1). Section 4.2 is devoted to definitions of some classes of functions needed throughout this study. In Section 4.3, we present and prove some important results that represent useful tools in proving our theorems. Section 4.4 is devoted to our main results on the asymptotic behavior of solutions for the fractional integro-differential equation (1.1) when the fractional derivatives are of Caputo type. This is followed by a section on similar results with the Riemann-Liouville fractional derivatives.

4.1 Well-posedness

There is fairly a large number of works on the well-posedness of fractional differential equations. In fact most of the analytical investigations are on existence and uniqueness. Several nonlinearities of the form

$$f(t,x),\ f(t,x,D^{\beta}_{0^+}x),\ f\left(t,x,D^{\beta}_{0^+}x,{}_0^t\, k(s,t,D^{\gamma}_{0^+}x(s))ds\right),$$

(with different kinds of fractional derivatives) or even more general have been treated. The local existence has been proved under weak conditions. For our purpose here, the local existence holds under the simple continuity of the nonlinearities (see[94, 96, 97, 98, 99, 100, 101, 104, 106, 107, 108, 110, 195]). Assumptions such as uniform boundedness, boundedness by continuous functions or by the product of functions of time in certain Lebesgue spaces times polynomials in the state, power type functions of the state or more generally increasing functions of the state (like our assumptions (4.14) and (4.15)) also have been used (see [99, 98, 97, 196, 103, 88, 107, 113]). As our interval of (local) existence is bounded then the continuity of the nonlinearities combined with appropriate underlying spaces (like our spaces (3.1), (4.7) and (4.65) here in the present dissertation) ensures the local existence of solutions. The initial data as well as the initial conditions are taken into account in the selection of the underlying space. To this end, several methods and techniques have been used: Shauder fixed point theorem, Shaefer fixed point theorem, Krasnoselskii's fixed point theorem, Banach contraction principle, Leray-Shauder alternative, upper, lower solutions, etc.

As for the integer order case, the uniqueness is proved under extra Lipschitz conditions on the nonlinearities (see [98, 196, 108, 195]).

In this dissertation we will be concerned mainly with asymptotic properties of solutions. Therefore, the local existence (which we will assume throughout this document) justifies our investigations. There is no need for uniqueness as our result will apply for all possible solutions. Unlike blow up in finite time, for the "nonexistence" results we shall not need any local existence result.

4.2 Notation and Abbreviations

Before presenting our results we need to define the following classes of functions:

Definition 4.1 *We say that a function* $h : [0,\infty) \to [0,\infty)$ *is of type* $\mathcal{H}_\sigma$ *if* $h \in C[0,\infty)$ *and* $t^\sigma h(t) \in L^1(1,\infty)$, $\sigma \geq 0$.

Definition 4.2 *We say that a function* $h : [0,\infty) \to [0,\infty)$ *is of type* $\mathcal{H}'_\sigma$ *if*

$h \in C[0,\infty)$ with $h' \in L^1(0,\infty)$ and

$$\int_1^t s^\sigma h'(t-s) l(s)\, ds \in L^1(1,\infty),$$

for some function l of type $\mathcal{H}_\sigma$.

Definition 4.3 *We say that a function $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of type $\mathcal{M}$ if it is continuous on $\{(t,s) : 0 \le s \le t \le \infty\}$ with $t^\alpha \max_{\tau \ge 1} h(\tau, t) \in L^1(1,\infty)$.*

Definition 4.4 *We say that a function g is of type $\mathcal{G}$ if it is continuous nondecreasing on $[0,\infty)$ and positive on $(0,\infty)$.*

Definition 4.5 *We say that a function g is of type $\mathcal{G}_1$ if it is of type $\mathcal{G}$ and $g(v) \le ug(\frac{v}{u})$, $u \ge 1, v > 0$.*

Definition 4.6 *We say that a function g is of type $\mathcal{G}_2$ if it is of type $\mathcal{G}_1$ and $\int_{t_0}^t \frac{d\tau}{g(\tau)} \to \infty$ as $t \to \infty$ for any $t_0 > 0$.*

The above classes are not empty. Examples showing this fact are given in sections 4.4 and 4.5 and their subsections.

4.3 Some Useful Lemmas

In this section, we prove some auxiliary lemmas needed in the next two sections. In the next lemma, we prove a version of Pinto's Lemmas 3.22 and 3.23, without the monotonicity and the ordering requirements.

Define the functions

$$\begin{aligned}\varphi_1(t) &:= \max_{s\in[0,t]}\left\{\omega_1(s)\right\}, \\ \varphi_i(t) &:= \max_{s\in[0,t]}\left\{\tfrac{\omega_i(s)}{\varphi_{i-1}(s)}\right\}\varphi_{i-1}(t),\ i=2,...,n.\end{aligned} \tag{4.1}$$

Note that $\varphi_i,\ i=1,...,n$ are nonnegative nondecreasing functions on $[0,\infty)$, $\omega_i(t)\leq\varphi_i(t)$, $i=1,...,n$ for all $t\in[0,\infty)$ and $\varphi_1\propto\varphi_2\propto...\propto\varphi_n$.

Lemma 4.1 *Let u, λ_i, $i=1,2$ be as in Lemma 3.22 and ω_i, $i=1,2$ be continuous functions on $[0,\infty)$ and positive on $(0,\infty)$. Assume further that c is a positive constant. If*

$$u(t)\leq c+\int_a^t\lambda_1(s)\omega_1(u(s))ds+\int_a^t\lambda_2(s)\omega_2(u(s))ds,\ t\in[a,b],$$

then, for $t\in[a,b_1]$,

$$u(t)\leq\Phi_2^{-1}\left(\Phi_2(c_1)+\int_a^t\lambda_2(s)ds\right), \tag{4.2}$$

where

1. $$\Phi_1(v)=\int_c^v\frac{d\tau}{\varphi_1(\tau)},\Phi_2(v)=\int_{v_0}^v\frac{d\tau}{\varphi_2(\tau)},\ v>0,\ v_0>0,$$

   φ_i are as in (4.1), $i=1,2$, and Φ_i^{-1} is the inverse function of Φ_i.

2. *The constants c_0 and c_1 are given by $c_0=c$ and $c_1=\Phi_1^{-1}\left(\int_a^{b_1}\lambda_1(s)ds\right)$.*

3. b_1 *is the largest number such that* $b_1 \geq a$ *and*

$$\int_a^{b_1} \lambda_i(s)ds \leq \int_{c_{i-1}}^{\infty} \frac{d\tau}{\varphi_i(\tau)}, \ i = 1, 2.$$

Proof. Let

$$v_1(t) = c + \int_a^t \lambda_1(s)\omega_1(u(s))ds, v_2(t) = \int_a^t \lambda_2(s)\omega_2(u(s))ds, \ t \in [a,b],$$

and $v = v_1 + v_2$. Clearly, $u \leq v$ and

$$\begin{aligned} v'(t) &= \lambda_1(t)\omega_1(u(t)) + \lambda_2(t)\omega_2(u(t) \leq \lambda_1(t)\varphi_1(u(t)) + \lambda_2(t)\varphi_2(u(t)) \\ &\leq \lambda_1(t)\varphi_1(v(t)) + \lambda_2(t)\varphi_2(v(t)), \quad t \in [a,b]. \end{aligned}$$

Then,

$$\frac{d}{dt}(\Phi_1(v(t))) = \frac{v'(t)}{\varphi_1(v(t))} \leq \lambda_1(t) + \lambda_2(t)\psi(v(t)), \tag{4.3}$$

where $\psi(v(t)) = \frac{\varphi_2(v(t))}{\varphi_1(v(t))}$ is nonnegative nondecreasing function on $[0, \infty)$.

Integrating (4.3) over $[a,t]$, $t \leq b_1$, yields

$$\begin{aligned} \Phi_1(v(t)) &\leq \int_a^t \lambda_1(s)ds + \int_a^t \lambda_2(s)\psi(v(s))\,ds \\ &\leq a_1 + \int_a^t \lambda_2(s)\psi(v(s))\,ds, \end{aligned}$$

where $\Phi_1(v(a)) = 0$ and $a_1 = \int_a^{b_1} \lambda_1(s)ds$.

Let $y(t) = \Phi_1(v(t))$, then

$$y(t) \leq a_1 + \int_a^t \lambda_2(s)\psi\left(\Phi_1^{-1}(y(s))\right) ds,\ t \in [a, b_1],$$

and using Bihari's inequality (Lemma 3.21), we obtain

$$y(t) \leq \Psi^{-1}\left(\Psi(a_1) + \int_a^t \lambda_2(s)ds\right),\ t \in [a, b_1],$$

where $\Psi(u) = \int_{c_2}^{u} \frac{d\tau}{\psi\left(\Phi_1^{-1}(\tau)\right)}$, $c_2 = \Phi_1(v_0)$. Now, as $u \leq v = \Phi_1^{-1}(y)$, it appears that

$$u(t) \leq \Psi^{-1}\left(\Psi(a_1) + \int_a^t \lambda_2(s)ds\right),\ t \in [a, b_1]. \tag{4.4}$$

In view of

$$\Psi(u) = \int_{c_2}^{u} \frac{d\tau}{\psi\left(\Phi_1^{-1}(\tau)\right)} = \int_{\Phi_1^{-1}(c_2)}^{\Phi_1^{-1}(u)} \frac{\Phi_1'(s)\, ds}{\psi(s)} = \int_{v_0}^{\Phi_1^{-1}(u)} \frac{ds}{\varphi_2(s)},$$

we entail that $\Psi = \Phi_2 \circ \Phi_1^{-1}$ and (4.4) imply (4.2). ∎

Using Lemma 4.1 and induction on n, we can prove the following lemma which improves Lemma 3.23.

Lemma 4.2 *Let $u, \lambda_i, i = 1, ..., n$ be continuous and nonnegative functions on $I = [a, b]$ and the functions ω_i, $i = 1, ..., n$ be continuous functions on $[0, \infty)$ and positive on $(0, \infty)$. Assume that c is a positive constant. If*

$$u(t) \leq c + \sum_{i=1}^{n} \int_a^t \lambda_i(s)\omega_i(u(s))ds,\ t \in [a, b],$$

then, for $t \in [a, b_1]$,

$$u(t) \leq \Phi_n^{-1}\left(\Phi_n(c_{n-1}) + \int_a^t \lambda_n(s)ds\right),$$

where

1. $\Phi_i(v) = \int_{v_i}^v \frac{d\tau}{\varphi_i(\tau)}, v > 0, v_i > 0, \varphi_i$ *are as in* (4.1), $i = 1, ..., n$ *and* Φ_i^{-1} *is the inverse function of* Φ_i.

2. *The constants* c_i *are given by* $c_0 = c$ *and* $c_i = \Phi_i^{-1}\left(\Phi_i(c_{i-1}) + \int_a^{b_1} \lambda_i(s)ds\right)$, $i = 1, ..., n-1$.

3. *The number* $b_1 \in [a, b]$ *is the largest number such that*

$$\int_a^{b_1} \lambda_i(s)ds \leq \int_{c_{i-1}}^{\infty} \frac{d\tau}{\varphi_i(\tau)}, i = 1, ..., n.$$

We will need to deal with the limit of the ratio of the Riemann-Liouville fractional integral $I_{a^+}^{\alpha+1}$ of a function and the power function t^α as $t \to \infty$. This is treated in the next lemma.

Lemma 4.3 *Let* $f \in L^1(a, \infty)$, $a \geq 0$. *Suppose that* u *and* v *are real-valued functions defined on* $[a, \infty)$, *then*

$$\lim_{t\to\infty} \frac{1}{t^\alpha} \int_a^t (t-s)^\alpha f(s, u(s), v(s))\, ds = \int_a^\infty f(s, u(s), v(s))\, ds\,.$$

Proof. It is enough to prove that

$$\lim_{t\to\infty}\left|\frac{1}{t^{\alpha}}\int_a^t (t-s)^{\alpha} f\left(s,u(s),v\left(s\right)\right)d\tau)-\frac{1}{\Gamma(\alpha+1)}\int_a^{\infty} f\left(s,u(s),v\left(s\right)\right)ds\right|=0.$$

Notice that

$$\begin{aligned}
&\left|\frac{1}{t^{\alpha}}\int_a^t (t-s)^{\alpha} f\left(s,u(s),v\left(s\right)\right)-\int_a^{\infty} f\left(s,u(s),v\left(s\right)\right)ds\right| \\
&=\left|\int_a^t (1-\frac{s}{t})^{\alpha} f\left(s,u(s),v\left(s\right)\right)ds-\int_a^{\infty} f\left(s,u(s),v\left(s\right)\right)ds\right| \\
&=\frac{1}{\Gamma(\alpha+1)}\left|\int_a^{\infty}\left(\chi_{[a,t]}\left(s\right)\left(1-\frac{s}{t}\right)^{\alpha}-1\right) f\left(s,u(s),v\left(s\right)\right)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha+1)}\int_a^{\infty}\left|\chi_{[a,t]}\left(s\right)\left(1-\frac{s}{t}\right)^{\alpha}-1\right|\left|f\left(s,u(s),v\left(s\right)\right)\right|ds,
\end{aligned}$$

where

$$\chi_{[a,t]}\left(s\right)=\begin{cases} 1, & s\in[a,t] \\ 0, & s\notin[a,t] \end{cases}.$$

As

$$\lim_{t\to\infty}\chi_{[a,t]}\left(s\right)\left(1-\frac{s}{t}\right)^{\alpha}=1,\ \text{for } s<t,$$

we obtain by the Dominated Convergence Theorem [193],

$$\begin{aligned}
&\lim_{t\to\infty}\left|\frac{1}{t^{\alpha}}\int_a^t (t-s)^{\alpha} f\left(s,u(s),v\left(s\right)\right)-\frac{1}{\Gamma(\alpha+1)}\int_a^{\infty} f\left(s,u(s),v\left(s\right)\right)ds\right| \\
&\leq \lim_{t\to\infty}\int_a^{\infty}\left|\chi_{[a,t]}\left(s\right)\left(1-\frac{s}{t}\right)^{\alpha}-1\right|\left|f\left(s,u(s),v\left(s\right)\right)\right|ds \\
&=\int_a^{\infty}\lim_{t\to\infty}\left|\chi_{[a,t]}\left(s\right)\left(1-\frac{s}{t}\right)^{\alpha}-1\right|\left|f\left(s,u(s),v\left(s\right)\right)\right|ds=0,
\end{aligned}$$

which is the desired result. ∎

4.4 Equations with Caputo Fractional Derivatives

In this section, we consider the following special case of (1.1),

$$\left({}^{C}D_{0^+}^{\alpha}x\right)'(t) = f\left(t, ({}^{C}D_{0^+}^{\beta}x)(t), \int_0^t k\left(t, s, \left({}^{C}D_{0^+}^{\gamma}x\right)(s)\right) ds\right),\ t \geq 0, \tag{4.5}$$

subject to

$$x(0) = c_0, \quad \left({}^{C}D_{0^+}^{\alpha}x\right)(0^+) = c_1, \quad c_0, c_1 \in \mathbb{R}, \tag{4.6}$$

where $0 \leq \beta \leq \alpha < 1$ and $0 \leq \gamma \leq \alpha < 1$. The functions f and k satisfy the following hypotheses:

(H$_1$) $f(t, u, v)$ is a continuous function in $D = \{(t, u, v) : t \geq 0,\ u,\ v \in \mathbb{R}\}$.

(H$_2$) $k(t, s, u)$ is continuous in $E = \{(t, s, u) : 0 \leq s < t < \infty,\ u \in \mathbb{R}\}$.

Definition 4.7 *We mean by a solution x of* $(4.5)-(4.6)$*, a function $x : [0, \infty) \to \mathbb{R}$, that it is continuable (continuous defined on $[0, \infty)$), satisfies the equation* (4.5) *and the initial conditions* (4.6) *and is in the space $C^{\alpha,1}[0, \infty)$ defined by*

$$C^{\alpha,1}[0, \infty) = \left\{x \in C[0, \infty),\ \left({}^{C}D_{0^+}^{\alpha}x\right)' \in C[0, \infty)\right\}. \tag{4.7}$$

Throughout this section and its subsections wherever we mention the solutions of the problem $(4.5)-(4.6)$ and its special forms, we mean the classical solutions in the sense of Definition 4.7.

In this section, we investigate the asymptotic behavior of solutions of (4.5) − (4.6), but before that we need the following two lemmas.

Lemma 4.4 *Let x be a solution of problem* (4.5)−(4.6) *with $f \in L^1(0,\infty)$. Then,*

$$\begin{aligned}\lim_{t\to\infty}\frac{x(t)}{t^\alpha} &= \lim_{t\to\infty}\frac{\left({}^C D^\alpha_{0^+}x\right)(t)}{\Gamma(\alpha+1)}\\ &= \frac{1}{\Gamma(\alpha+1)}\left(c_1+\int_0^\infty f\left(s,({}^C D^\beta_{0^+}x)(s),\int_0^s k(s,\tau,\left({}^C D^\gamma_{0^+}x\right)(\tau))d\tau\right)ds\right).\end{aligned}$$

Proof. Integrating both sides of (4.5) over $[0,t]$ yields

$$\left({}^C D^\alpha_{0^+}x\right)(t) = c_1+\int_0^t f\left(s,({}^C D^\beta_{0^+}x)(s),\int_0^s k\left(s,\tau,\left({}^C D^\gamma_{0^+}x\right)(\tau)\right)d\tau\right)ds. \quad (4.8)$$

Applying $I^\alpha_{a^+}$ to both sides of equation (4.8), taking into account the semigroup property (Lemma (3.10)) and Lemmas 3.17 and 3.5, we get

$$x(t) = c_0+\frac{c_1 t^\alpha}{\Gamma(\alpha+1)}+\left(I^{\alpha+1}_{0^+}f\left(s,({}^C D^\beta_{0^+}x)(s),\int_0^s k(s,\tau,\left({}^C D^\gamma_{0^+}x\right)(\tau))d\tau\right)\right)(t). \quad (4.9)$$

Next, we divide both sides of (4.9) by t^α, to obtain for $t>0$,

$$\frac{x(t)}{t^\alpha} = \frac{c_0}{t^\alpha}+\frac{c_1}{\Gamma(\alpha+1)}+\frac{1}{t^\alpha}\left(I^{\alpha+1}_{0^+}f\left(s,\left({}^C D^\beta_{0^+}x\right)(s),\int_0^s k(s,\tau,\left({}^C D^\gamma_{0^+}x\right)(\tau))d\tau\right)\right)(t).$$

By taking the limit as $t\to\infty$, the result follows from (4.8) and Lemma 4.3. ∎

Lemma 4.5 *If x is a solution for the problem* (4.5) − (4.6) *and the functions f*

and k satisfy ($\boldsymbol{H}_1$) and ($\boldsymbol{H}_2$), then

$$\frac{|x(t)|}{t^\alpha} \leq A_1 + A_2 \int_0^t \left| f\left(s, ({}^C D_{0^+}^\beta x)(s), \int_0^s k(s, \tau, \left({}^C D_{0^+}^\gamma x\right)(\tau))d\tau\right)\right| ds, \quad (4.10)$$

for all $t \geq 1$, where

$$A_1 = |c_0| + \frac{|c_1|}{\Gamma(\alpha+1)}, \quad A_2 = \frac{1}{\Gamma(\alpha+1)}.$$

Proof. It follows from (4.9) that

$$\begin{aligned} |x(t)| &\leq |c_0| + \frac{|c_1|\, t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \left(I_{0^+}^{\alpha+1} \left| f\left(s, ({}^C D_{0^+}^\beta x)(s), \int_0^s k(s, \tau, \left({}^C D_{0^+}^\gamma x\right)(\tau))d\tau\right)\right|\right)(t) \\ &\leq |c_0| + \frac{1}{\Gamma(\alpha+1)} \Big(|c_1|\, t^\alpha \\ &\quad + \int_0^t (t-s)^\alpha \left| f\left(s, ({}^C D_{0^+}^\beta x)(s), \int_0^s k(s, \tau, \left({}^C D_{0^+}^\gamma x\right)(\tau))d\tau\right)\right| ds\Big) \\ &\leq |c_0| + \frac{|c_1|\, t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \frac{t^\alpha}{\Gamma(\alpha+1)} \int_0^t \left| f\left(s, ({}^C D_{0^+}^\beta x)(s), \int_0^s k(s, \tau, \left({}^C D_{0^+}^\gamma x\right)(\tau))d\tau\right)\right| ds. \quad (4.11) \end{aligned}$$

for all $t > 0$. Dividing by t^α, $t \geq 1$, completes the proof. ∎

The rest of this section is divided into five subsections organized according to the types of conditions on the nonlinear function f and the kernel k.

4.4.1 Case of a Non-fractional Source

In this subsection, we consider the problem $(4.5) - (4.6)$ with $\beta = \gamma = 0$ and $0 < \alpha < 1$, that is,

$$\left({}^{C}D_{0^+}^{\alpha}x\right)'(t) = f\left(t, x(t), \int_0^t k\left(t, s, x(s)\right) ds\right), \ t \geq 0, \tag{4.12}$$

subject to

$$x(0) = c_0, \quad \left({}^{C}D_{0^+}^{\alpha}x\right)(0^+) = c_1, \quad c_0, c_1 \in \mathbb{R}. \tag{4.13}$$

In addition to the conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, the functions f and k are assumed here to satisfy the hypotheses

$(\mathbf{H}_3)$ There are functions h_1, h_3 of type $\mathcal{H}_\alpha$, h_2 of type $\mathcal{H}_0$, and g_i of type $\mathcal{G}_2$, $i = 1, 2, 3$ with $g_1 \propto g_2 \propto g_3$ such that

$$|f(t, u, v)| \leq h_1(t) g_1(|u|) + h_2(t) g_2(|v|), \quad (t, u, v) \in D, \tag{4.14}$$

$$|k(t, s, u)| \leq h_3(s) g_3(|u|), \ (t, s, u) \in E. \tag{4.15}$$

A nonlinear integral inequality for several nonlinearities is obtained in the following lemma.

Lemma 4.6 *Let A_1 and A_2 be positive constants, and*

$$\begin{aligned} u_1(t) &= A_1 + A_2 \int_0^t h_1(s) g_1(|x(s)|) ds, \\ u_2(t) &= A_2 \int_0^t h_2(s) g_2(u_3(s)) ds, \\ u_3(t) &= \int_0^t h_3(s) g_3(|x(s)|) ds, \text{ for all } t \geq 0, \end{aligned} \tag{4.16}$$

where h_1, h_3 are of type $\mathcal{H}_\alpha$, h_2 is of type $\mathcal{H}_0$ and g_i are of type $\mathcal{G}_2$, $i = 1, 2, 3$ and x is a solution for the problem $(4.12) - (4.13)$. Then,

$$\begin{aligned} u(t) \leq\ & u(1) + A_2 \int_1^t s^\alpha h_1(s) g_1(u(s)) ds + A_2 \int_1^t h_2(s) g_2(u(s)) ds \\ & + \int_1^t s^\alpha h_3(s) g_3(u(s)) ds, \text{ for all } t \geq 1, \end{aligned} \tag{4.17}$$

where

$$u = u_1 + u_2 + u_3, \tag{4.18}$$

and

$$\begin{aligned} u(1) =\ & A_1 + A_2 \int_0^1 h_1(s) g_1(|x(s)|) ds + A_2 \int_0^1 h_2(s) g_2(u_3(s)) ds \\ & + \int_0^1 h_3(s) g_3(|x(s)|) ds. \end{aligned} \tag{4.19}$$

Proof. We start by differentiating the expression u defined in (4.18), to obtain

$$u'(t) = A_2 h_1(t) g_1(|x(t)|) + A_2 h_2(t) g_2(u_3(t)) + h_3(t) g_3(|x(t)|), \ t > 0.$$

Since

$$
\begin{aligned}
g_1(|x(t)|) &\leq t^{\alpha} g_1\left(\frac{|x(t)|}{t^{\alpha}}\right), \\
g_3\left(|x(t)|\right) &\leq t^{\alpha} g_3\left(\frac{|x(t)|}{t^{\alpha}}\right), \quad \text{for all} \quad t \geq 1,
\end{aligned}
$$

we see that, for all $t \geq 1$,

$$
u'\left(t\right) \leq A_2 t^{\alpha} h_1(t) g_1\left(\frac{|x(t)|}{t^{\alpha}}\right) + A_2 h_2(t) g_2(u_3\left(t\right)) + t^{\alpha} h_3(t) g_3\left(\frac{|x(t)|}{t^{\alpha}}\right). \tag{4.20}
$$

Let

$$
\begin{aligned}
z(t) &= A_1 + A_2 \int_0^t h_1(s) g_1(|x(s)|) ds \\
&\quad + A_2 \int_0^t h_2(s) g_2\left(\int_0^s h_3(\tau) g_3(|x(\tau)|) d\tau\right) ds, \ t \geq 0,
\end{aligned} \tag{4.21}
$$

It is clear from Lemma 4.5 and the condition $(\mathbf{H}_3)$ that

$$
\frac{|x(t)|}{t^{\alpha}} \leq z(t) \text{ for all } t \geq 1. \tag{4.22}
$$

Also, we have

$$
z(t) \leq u\left(t\right) \text{ and } u_3\left(t\right) \leq u\left(t\right), \quad \text{for all} \quad t \geq 0. \tag{4.23}
$$

So for all $t \geq 1$,

$$u'(t) \leq A_2 t^{\alpha} h_1(t) g_1(u(t)) + A_2 h_2(t) g_2(u(t)) + t^{\alpha} h_3(t) g_3(u(t)). \tag{4.24}$$

Integrating both sides of (4.24) over $[1, t]$ leads to

$$\begin{aligned} u(t) &\leq u(1) + A_2 \int_1^t s^{\alpha} h_1(s) g_1(u(s)) ds + A_2 \int_1^t h_2(s) g_2(u(s)) ds \\ &\quad + \int_1^t s^{\alpha} h_3(s) g_3(u(s)) ds, \end{aligned}$$

where $u(1)$ can be expressed using (4.18) as given in (4.19). ∎

The main result of this section is given in the following theorem.

Theorem 4.1 *Suppose that the functions f and k satisfy ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_3$). Then, any solution of the problem* (4.12)−(4.13) *is asymptotic to bt^{α} when $t \to \infty$, for some $b \in \mathbb{R}$.*

Proof. Direct application of Lemma 3.23 to (4.17) (see Lemma 4.6) yields for $t \geq 1$,

$$u(t) \leq G_3^{-1}\left(G_3(C_2) + \int_1^t s^{\alpha} h_3(s) ds\right),$$

where

$$G_i(v) = \int_{v_i}^{v} \frac{d\sigma}{g_i(\sigma)}, \ v > 0, \ v_i > 0, \ i = 1, 2, 3,$$

and G_i^{-1} is the inverse function of G_i,

$$\begin{aligned} C_0 &= A_4,\ C_1 = G_1^{-1}\left(G_1(C_0) + \int_1^\infty s^\alpha h_1(s)ds\right), \\ C_2 &= G_2^{-1}\left(G_2(C_1) + \int_1^\infty h_2(s)ds\right). \end{aligned}$$

Note that the hypotheses of Lemma 3.23 are all satisfied with $b_1 = \infty$ because $G_i(\infty) = \int_{t_0}^\infty \frac{d\sigma}{g_i(\sigma)} = \infty$, $i = 1, 2, 3$, for any $t_0 > 0$.

As $\int_1^\infty s^\alpha h_1(s)ds < \infty$, $\int_1^\infty h_2(s)ds < \infty$, $\int_1^\infty s^\alpha h_3(s)ds < \infty$, G_i^{-1} is nondecreasing, $i = 1, 2, 3$, we see that

$$G_1(C_0) + \int_1^\infty s^\alpha h_1(s)ds < \infty,\ G_2(C_1) + \int_1^\infty h_2(s)ds < \infty,$$

$$G_3^{-1}\left(G_3(C_2) + \int_1^\infty s^\alpha h_3(s)ds\right) < \infty,$$

and

$$u(t) \leq A_5 \ \text{ for some constant } \ A_5 > 0. \tag{4.25}$$

Since

$$\frac{|x(t)|}{t^\alpha} \leq u(t) \ \text{ for all } \ t \geq 1, \tag{4.26}$$

we conclude that

$$\frac{|x(t)|}{t^\alpha} \leq A_5, \ \text{ for all } \ t \geq 1. \tag{4.27}$$

Now,

$$
\begin{aligned}
&\int_0^t \left| f\left(s, x(s), \int_0^s k\left(s,\tau,x(\tau)\right) d\tau\right)\right| ds \\
&\leq \int_0^t \left[h_1(s)g_1(|x(s)|) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(|x(\tau)|)d\tau\right)\right] ds \\
&\leq \int_0^1 \left[h_1(s)g_1(|x(s)|) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(|x(\tau)|)d\tau\right)\right] ds \\
&+ \int_1^t \left[s^\alpha h_1(s)g_1\left(\frac{|x(s)|}{s^\alpha}\right) + h_2(s)g_2\left(u\left(s\right)\right)\right] ds = I_1 + I_2\,,
\end{aligned}
$$

where

$$
I_1 = \int_0^1 \left[h_1(s)g_1(|x(s)|) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(|x(\tau)|)d\tau\right)\right] ds
$$

is finite by the continuity of the integrand functions over $[0,1]$. Also,

$$
I_2 = \int_1^t \left[s^\alpha h_1(s)g_1\left(\frac{|x(s)|}{s^\alpha}\right) + h_2(s)g_2\left(u\left(s\right)\right)\right] ds
$$

is uniformly bounded by (4.25) and (4.27).

Hence the integral

$$
\int_0^t f\left(s, x(s), \int_0^s k\left(s,\tau,x(\tau)\right) d\tau\right) ds
$$

is absolutely convergent and

$$
\lim_{t\to\infty} \int_0^t f\left(s, x(s), \int_0^s k\left(s,\tau,x(\tau)\right) d\tau\right) ds < \infty,
$$

which in view of (4.8) in the proof of Lemma 4.3 implies that there exists a real number b such that

$$\lim_{t\to\infty} \frac{\left({}^{C}D_{0+}^{\alpha}x\right)(t)}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)}\left(c_1 + \int_0^\infty f\left(s, x(s), \int_0^s k\left(s,\tau,x(\tau)\right)d\tau\right)ds\right) = b.$$

Finally, we deduce from Lemma 4.4 that

$$\lim_{t\to\infty}\frac{x(t)}{t^\alpha} = b,$$

and the proof is complete. ∎

Example 4.1 *Consider the fractional integro-differential equation*

$$\left({}^{C}D_{0+}^{\alpha}x\right)'(t) = t^{\mu_1}e^{-t}\ \left(|x(t)|\right)^{\lambda_1} + t^{\mu_2}e^{-t}\left(\int_0^t s^{\mu_3}e^{-(s+t)}\left(|x(s)|\right)^{\lambda_3}ds\right)^{\lambda_2},\ t>0, \tag{4.28}$$

where

$$0<\alpha<1,\ 0\le\lambda_1\le\lambda_2\le\lambda_3\le 1 \text{ and } \mu_1,\mu_2,\mu_3>0.$$

Let

$$h_i(t) = t^{\mu_i}e^{-\rho_i t},\ g_i(t) = t^{\lambda_i},\quad 0<\rho_i\le 1,\ i=1,2,3,$$

Obviously, the conditions ($\boldsymbol{H}_1$)*,* ($\boldsymbol{H}_2$) *and* ($\boldsymbol{H}_3$) *are satisfied*

$$\begin{aligned}
&\int_1^{\infty} t^{\alpha} h_1(t)\,dt < \int_0^{\infty} t^{\alpha} h_1(t)\,dt = \int_0^{\infty} t^{\alpha+\mu_1} e^{-\rho_1 t} dt = \frac{\Gamma(\alpha+\mu_1+1)}{\rho_1^{\alpha+\mu_1+1}} < \infty,\\
&\int_1^{\infty} h_2(t)\,dt < \int_0^{\infty} h_2(t)\,dt = \int_0^{\infty} t^{\mu_2} e^{-\rho_2 t} dt = \frac{\Gamma(\mu_2+1)}{\rho_2^{\mu_2+1}} < \infty,\\
&\int_1^{\infty} t^{\alpha} h_3(t)\,dt < \int_0^{\infty} t^{\alpha} h_3(t)\,dt = \int_0^{\infty} t^{\alpha+\mu_2} e^{-\rho_3 t} dt = \frac{\Gamma(\alpha+\mu_2+1)}{\rho_3^{\alpha+\mu_2+1}} < \infty,\\
&\int_{t_0}^{\infty} \frac{dt}{g_i(t)} = \int_{t_0}^{\infty} \frac{dt}{t^{\lambda_i}} = \infty,\ \ g_i(v) = v^{\lambda_i} \le u^{1-\lambda_i} v^{\lambda_i} = u g_i\left(\frac{v}{u}\right),\ u \ge 1,\ \ v, t_0 > 0.
\end{aligned}$$

All the conditions of Theorem 4.1 are fulfilled. We conclude that every solution of the problem (4.28) $-$ (4.13) *is asymptotic to* $d_1 t^{\alpha}$ *when* $t \to \infty$*,* $d_1 \in \mathbb{R}$*.*

Remark 4.1 *Theorem 4.1 and Lemma 4.6 can be proved without the monotonicity condition of the functions* g_i *and the condition* $g_1 \propto g_2 \propto g_3$ *as we will show in the next results.*

Consider the following hypotheses:

($\mathbf{H}_4$) There are functions h_1, h_3 of type $\mathcal{H}_{\alpha}$, h_2 of type $\mathcal{H}_0$, and g_i of type $\mathcal{G}_1$, $i = 1, 2, 3$ such that (4.14) and (4.15) are satisfied and

$$\int_{t_0}^{t} \frac{d\tau}{\varphi_i(\tau)} \to \infty \ \text{ as } t \to \infty, t_0 > 0,$$

where

$$\begin{aligned}
&\varphi_1(t) = \max_{s\in[0,t]} \{g_1(s)\},\ \varphi_2(t) = \max_{s\in[0,t]} \left\{\tfrac{g_2(s)}{\varphi_1(s)}\right\} \varphi_1(t),\\
&\varphi_3(t) = \max_{s\in[0,t]} \left\{\tfrac{g_3(s)}{\varphi_2(s)}\right\} \varphi_2(t),\ t \ge 0.
\end{aligned} \tag{4.29}$$

The fact that $g_i(t) \leq \varphi_i(t)$, $i = 1, 2, 3$ for all $t \in [0, \infty)$ clearly leads, in view of Lemma 4.6, to the following result.

Lemma 4.7 *Suppose that x is a solution for the problem* (4.12) – (4.13)*. Let u, u_i, h_i be as in Lemma 4.6 and let g_i be as in ($\boldsymbol{H}_4$), $i = 1, 2, 3$. Then,*

$$\begin{aligned} u(t) \quad \leq \quad & u(1) + A_2 \int_1^t s^{\alpha} h_1(s) \varphi_1(u(s)) ds + A_2 \int_1^t h_2(s) \varphi_2(u(s)) ds \\ & + \int_1^t s^{\alpha} h_3(s) \varphi_3(u(s)) ds, \quad \textit{for all } t \geq 1, \end{aligned} \tag{4.30}$$

where φ_i, $i = 1, 2, 3$ are as in (4.29).

Note that applying Lemma 3.23 to (4.30) gives

$$u(t) \leq \Phi_3^{-1}\left(\Phi_3(C_2) + \int_1^t s^{\alpha} h_3(s) ds\right), \quad t \geq 1,$$

where

$$\Phi_i(v) = \int_{v_i}^{v} \frac{d\sigma}{\varphi_i(\sigma)}, \;\; v > 0, \;\; v_i > 0, \;\; i = 1, 2, 3, \tag{4.31}$$

$\varphi_i, i = 1, 2, 3$ are as in (4.29) and Φ_i^{-1} is the inverse function of Φ_i,

$$\begin{gathered} C_0 = A_4, \;\; C_1 = \Phi_1^{-1}\left(\Phi_1(C_0) + \textstyle\int_1^{\infty} s^{\alpha} h_1(s) ds\right), \\ C_2 = \Phi_2^{-1}\left(\Phi_2(C_1) + \textstyle\int_1^{\infty} h_2(s) ds\right). \end{gathered} \tag{4.32}$$

Therefore, the following result can be proved in the same way as Theorem 4.1.

Theorem 4.2 *Suppose that the functions f and k satisfy the conditions ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_4$). Then, any solution of the problem* (4.12) – (4.13) *is asymptotic* *to* bt^{α} *when* $t \to \infty$, *for some* $b \in \mathbb{R}$.

4.4.2 Case of a Bounded Non-convolution Kernel

We investigate, in this subsection, the asymptotic behavior of solutions of (4.12)−(4.13) when the functions f and k satisfy the following hypotheses

(H$_5$) There are functions h_1 of type $\mathcal{H}_{\alpha}$, h_2 of type $\mathcal{H}_0$, h of type $\mathcal{M}$ and g_i of type $\mathcal{G}_1$, $i = 1, 2, 3$, such that the functions f, k and $\varphi_i, i = 1, 2, 3$, satisfy (4.14),

$$|k(t, s, u)| \leq h(t, s) g_3(|u|), \ (t, s, u) \in E, \tag{4.33}$$

and

$$\int_{t_0}^{t} \frac{d\tau}{\varphi_i(\tau)} \to \infty \text{ as } t \to \infty, t_0 > 0, \tag{4.34}$$

where φ_i, $i = 1, 2, 3$ are given in (4.29).

An example of the function h of type $\mathcal{M}$, in (**H$_5$**), is $h(t, s) = se^{-(s+t)}$ with

$$\int_1^{\infty} t^{\alpha} \max_{\tau \geq 1} h(\tau, t) dt = \int_1^{\infty} e^{-1} t^{\alpha+1} e^{-t} dt \leq e^{-1} \int_0^{\infty} t^{\alpha+1} e^{-t} dt = e^{-1} \Gamma(\alpha + 2) < \infty.$$

Lemma 4.8 *Let* u *be a continuous and nonnegative function on* $[1, \infty)$ *and let* c *be a positive constant. Assume that*

$$\begin{aligned} u(t) \ &\leq \ c + \int_1^t s^{\alpha} h_1(s) \varphi_1(u(s)) ds + \int_1^t h_2(s) \varphi_2(u(s)) ds \\ &\quad + \int_1^t s^{\alpha} h(t, s) \varphi_3(u(s)) \, ds, \ \ \textit{for all } t \geq 1, \end{aligned}$$

where $h_1, h_2,$ h and φ_i, $i = 1, 2, 3,$ satisfy ($\boldsymbol{H}_5$). Then,

$$u(t) \leq \Phi_3^{-1}\left(\Phi_3(C_2) + \int_1^t s^\alpha \max_{1\leq\tau\leq t} h(\tau, s)ds\right), \tag{4.35}$$

where Φ_i, Φ_i^{-1} and C_i are as in (4.31) *and* (4.32).

Proof. Let $\tilde{h}(t,s) = \max_{1\leq\tau\leq t} h(\tau, s)$. Clearly, $\tilde{h}(t,s)$ is nonnegative and nondecreasing in t for each fixed s and $h(t,s) \leq \tilde{h}(t,s)$ for all $1 \leq s \leq t$. Let

$$\begin{aligned} y(t) &= c + \int_1^t s^\alpha h_1(s)\varphi_1(u(s))ds + \int_1^t h_2(s)\varphi_2(u(s))ds \\ &\quad + \int_1^t s^\alpha \tilde{h}(T,s)\varphi_3(u(s))\,ds, \ 1 \leq t \leq T, \end{aligned}$$

where T is arbitrarily chosen such that $1 \leq T \leq \infty$. Clearly, $u(t) \leq y(t)$ and

$$\begin{aligned} y(t) &\leq c + \int_1^t s^\alpha h_1(s)\varphi_1(y(s))ds + \int_1^t h_2(s)\varphi_2(y(s))ds \\ &\quad + \int_1^t s^\alpha \tilde{h}(T,s)\varphi_3(y(s))\,ds, \ 1 \leq t \leq T. \end{aligned} \tag{4.36}$$

The result follows by applying Lemma 3.23 to (4.36) because T is arbitrarily chosen. ∎

The main result of this subsection is given next.

Theorem 4.3 *Suppose that the functions f and k satisfy ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_5$). Then, any solution of the problem* (4.12)−(4.13) *is asymptotic to bt^α when $t \to \infty$, for some $b \in \mathbb{R}$.*

Proof. We know from the condition $(\mathbf{H}_5)$ that $t^{\alpha}h_1(t), h_2(t),\ t^{\alpha}\max\limits_{\tau\geq 1} h(\tau,t)\in L^1(1,\infty)$, Φ_i^{-1} is nondecreasing and $\Phi_i(\infty)=\infty$, $i=1,2,3$, thus

$$\Phi_1(C_0)+\int_1^{\infty} s^{\alpha}h_1(s)ds<\infty,\ \Phi_2(C_1)+\int_1^{\infty} h_2(s)ds<\infty,$$

$$\Phi_3(C_2)+\int_1^{\infty} s^{\alpha}\max_{\tau\geq 1} h(\tau,s)ds<\infty.$$

Consequently, there exists a constant $B_5>0$ such that (4.35) implies that

$$u(t)\leq B_5 \text{ for all } \ t\geq 1, \tag{4.37}$$

and

$$\frac{|x(t)|}{t^{\alpha}}\leq u(t)\leq B_5, \quad \text{for all} \ \ t\geq 1. \tag{4.38}$$

Therefore,

$$\begin{aligned}
&\int_0^t \left| f\left(s, x(s), \int_0^s k\left(s,\tau,x(\tau)\right)d\tau\right)\right| ds\\
&\leq \int_0^1 \left(h_1(s)g_1(|x(s)|)+h_2(s)g_2\left(\int_0^s h(s,\tau)g_3(|x(\tau)|)d\tau\right)\right)ds\\
&+\int_1^t \left(s^{\alpha}h_1(s)\varphi_1\left(\frac{|x(s)|}{s^{\alpha}}\right)+h_2(s)\varphi_2\left(u\left(s\right)\right)\right)ds.
\end{aligned}$$

As

$$\int_1^t \left(s^{\alpha}h_1(s)\varphi_1\left(\frac{|x(s)|}{s^{\alpha}}\right)+h_2(s)\varphi_2\left(u\left(s\right)\right)\right)ds$$

is uniformly bounded from (4.37) and (4.38), the integral

$$\int_0^t f\left(s, x(s), \int_0^s k\left(s, \tau, x(\tau)\right) d\tau\right) ds$$

is absolutely convergent and the rest of the proof is as in the proof of Theorem 4.2. ∎

4.4.3 Case of a Convolution Kernel

In this subsection, we discuss the asymptotic behavior of solutions of (4.12)−(4.13) when the functions f and k satisfy the following hypotheses

($\mathbf{H}_6$) There are functions h_1 of type $\mathcal{H}_\alpha$, h_2 of type $\mathcal{H}_0$, h of type $\mathcal{H}'_\alpha$ and g_i of type $\mathcal{G}_1$, $i = 1, 2, 3$ such that the functions f, k and $\varphi_i, i = 1, 2, 3$ satisfy (4.14) ,

$$|k(t, s, u)| \leq h(t-s)l\left(s\right) g_3(|u|), (t, s, u) \in E,$$

and (4.34), respectively.

Lemma 4.9 *Suppose that x is a solution for the problem* (4.12) − (4.13)*. Let u, u_1 and u_2 be as in Lemma 4.6 with*

$$u_3\left(t\right) = \int_0^t h(t-s)l\left(s\right) g_3(|x(s)|)ds,\ \textit{for all}\ t \geq 0,$$

where h_1, h_2, h and g_i, $i = 1, 2, 3$ satisfy (H_6). Then,

$$\begin{aligned} u(t) &\leq \tilde{A}_1 + A_2 \int_1^t s^\alpha h_1(s)\varphi_1(u(s))ds + A_2 \int_1^t h_2(s)\varphi_2(u(s))ds \\ &\quad + \int_1^t s^\alpha h(t-s) l(s) \varphi_3(u(s))\, ds, \quad \text{for all } t \geq 1, \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} \tilde{A}_1 &= A_1 + A_2 \int_0^1 h_1(s) g_1(|x(s)|)ds + A_2 \int_0^1 h_2(s) g_2(u_3(s))\, ds \\ &\quad + \int_0^1 \max_{0 \leq t < 1} h(t-s) l(s)\, g_3(|x(s)|)ds. \end{aligned} \tag{4.40}$$

and φ_i, $i = 1, 2, 3$ are as in (4.29).

Proof. For $u = u_1 + u_2 + u_3$, we see that

$$\begin{aligned} u(t) &= A_1 + A_2 \int_0^t h_1(s) g_1(|x(s)|)\, ds + A_2 \int_0^t h_2(s) g_2(u_3(s))\, ds \\ &\quad + \int_0^t h(t-s) l(s)\, g_3(|x(s)|)\, ds \\ &\leq \tilde{A}_1 + A_2 \int_1^t h_1(s) g_1(|x(s)|)\, ds + A_2 \int_1^t h_2(s) g_2(u_3(s))\, ds \\ &\quad + \int_1^t h(t-s) l(s)\, g_3(|x(s)|)\, ds, \end{aligned}$$

where $\tilde{A}_1$ is the constant given in (4.40). Let

$$\begin{aligned} z(t) &= A_1 + A_2 \int_0^t h_1(s) g_1(|x(s)|)ds \\ &\quad + A_2 \int_0^t h_2(s) g_2\left(\int_0^s h(s-\tau) l(\tau)\, g_3(|x(\tau)|)d\tau\right) ds, \ t \geq 0 \end{aligned} \tag{4.41}$$

Clearly,

$$u_3(t) \le u(t) \text{ and } z(t) \le u(t), \text{ for all } t \ge 0.$$

We have from Lemma 4.5 and the assumption ($\mathbf{H}_6$) that

$$\frac{|x(t)|}{t^\alpha} \le z(t) \text{ for all } t \ge 1.$$

Since g_1 and g_3 are of type $\mathcal{G}_1$ and $g_i \le \varphi_i$, $i = 1, 2, 3$, we deduce that

$$\begin{aligned} u(t) \quad \le \quad & \tilde{A}_1 + A_2 \int_1^t s^\alpha h_1(s)\varphi_1(u(s))\, ds + A_2 \int_1^t h_2(s)\varphi_2(u(s))\, ds \\ & + \int_1^t s^\alpha h(t-s) l(s)\, \varphi_3(u(s))\, ds. \end{aligned}$$

for all $t \ge 1$. ∎

Lemma 4.10 *Let u be a continuous and nonnegative function on $[1, \infty)$ and let c be a positive constant. Assume that*

$$\begin{aligned} u(t) \quad \le \quad & c + \int_1^t s^\alpha h_1(s)\varphi_1(u(s))ds + \int_1^t h_2(s)\varphi_2(u(s))ds \\ & + \int_1^t s^\alpha h(t-s) l(s)\, \varphi_3(u(s))\, ds, \quad \textit{for all } t \ge 1, \end{aligned}$$

where h_1, h_2, h and φ_i, $i = 1, 2, 3$ satisfy ($\boldsymbol{H}_6$). Then,

$$u(t) \le \Phi_3^{-1}\left(\Phi_3(C_2) + \int_1^t H(s)ds\right), \textit{ for all } t \ge 1, \tag{4.42}$$

where

$$H(t) = h(0)t^{\alpha}l\,(t) + t^{\alpha}l\,(t)\int_{1}^{\infty}|h'(s-1)|\,ds,\;\; t \geq 1,$$

and Φ_i^{-1}, Φ_i *and* C_i, $i = 1, 2, 3,$ *are as in 4.31 and 4.32.*

Proof. Let

$$\begin{aligned} y\,(t) &= c + \int_{1}^{t} s^{\alpha}h_1(s)\varphi_1(u\,(s))ds + \int_{1}^{t} h_2(s)\varphi_2(u\,(s))ds, \\ &\quad + \int_{1}^{t} s^{\alpha}h(t-s)l\,(s)\,\varphi_3\,(u\,(s))\,ds,\;\; t \geq 1. \end{aligned}$$

Then, $u\,(t) \leq y\,(t)$ for all $t \geq 1$, and

$$\begin{aligned} y'\,(t) &= t^{\alpha}h_1(t)\varphi_1(u\,(t)) + h_2(t)\varphi_2(u\,(t)) + t^{\alpha}h(0)l\,(t)\,\varphi_3\,(u\,(t)) \\ &\quad + \int_{1}^{t} s^{\alpha}h'(t-s)l\,(s)\,\varphi_3\,(u\,(s))\,ds \\ &\leq t^{\alpha}h_1(t)\varphi_1(y\,(t)) + h_2(t)\varphi_2(y\,(t)) + t^{\alpha}h(0)l\,(t)\,\varphi_3\,(y\,(t)) \\ &\quad + \int_{1}^{t} s^{\alpha}\,|h'(t-s)|\,l\,(s)\,\varphi_3\,(u\,(s))\,ds \\ &\leq t^{\alpha}h_1(t)\varphi_1(y\,(t)) + h_2(t)\varphi_2(y\,(t)) + t^{\alpha}h(0)l\,(t)\,\varphi_3\,(y\,(t)) \\ &\quad + \int_{1}^{t} s^{\alpha}\,|h'(t-s)|\,l\,(s)\,\varphi_3\,(y\,(s))\,ds. \end{aligned} \tag{4.43}$$

Suppose that

$$y_1\,(t) = \int_{0}^{t-1}|h'(s)|\left(\int_{t-s}^{t}\tau^{\alpha}l\,(\tau)\,\varphi_3\,(y\,(\tau))\,d\tau\right)ds,\;\; t \geq 1.$$

The derivative of y_1 takes the form

$$\begin{aligned}
y_1'(t) &= |h'(t-1)|\left(\int_1^t \tau^\alpha l(\tau)\varphi_3(y(\tau))\,d\tau\right) + \int_0^{t-1}|h'(s)|\left[t^\alpha l(t)\varphi_3(y(t))\right.\\
&\quad \left.-(t-s)^\alpha l(t-s)\varphi_3(y(t-s))\right]ds\\
&= |h'(t-1)|\int_1^t \tau^\alpha l(\tau)\varphi_3(y(\tau))\,d\tau + t^\alpha l(t)\varphi_3(y(t))\int_0^{t-1}|h'(s)|\,ds\\
&\quad -\int_0^{t-1}(t-s)^\alpha l(t-s)|h'(s)|\varphi_3(y(t-s))\,ds\\
&= |h'(t-1)|\int_1^t \tau^\alpha l(\tau)\varphi_3(y(\tau))\,d\tau + t^\alpha l(t)\varphi_3(y(t))\int_1^t|h'(\sigma-1)|\,d\sigma\\
&\quad -\int_1^t \sigma^\alpha l(\sigma)|h'(t-\sigma)|\varphi_3(y(\sigma))\,d\sigma,\ t\geq 1.
\end{aligned}$$

Now, setting

$$w(t) := y(t) + y_1(t),\ t \geq 1,$$

leads to

$$\begin{aligned}
w'(t) &= t^\alpha h_1(t)\varphi_1(u(t)) + h_2(t)\varphi_2(u(t)) + t^\alpha h(0)l(t)\varphi_3(u(t))\\
&\quad +\int_1^t s^\alpha h'(t-s)l(s)\varphi_3(u(s))\,ds + |h'(t-1)|\int_1^t \tau^\alpha l(\tau)\varphi_3(y(\tau))\,d\tau\\
&\quad +t^\alpha l(t)\varphi_3(y(t))\int_1^t|h'(\sigma-1)|\,d\sigma - \int_1^t \sigma^\alpha l(\sigma)|h'(t-\sigma)|\varphi_3(y(\sigma))\,d\sigma.
\end{aligned}$$

Clearly, $u(t) \leq y(t) \leq w(t)$ for all $t \geq 1$, so,

$$\begin{aligned}
w'(t) &\leq t^{\alpha}h_1(t)\varphi_1(y(t)) + h_2(t)\varphi_2(y(t)) + t^{\alpha}h(0)l(t)\varphi_3(y(t)) \\
&\quad + \int_1^t s^{\alpha}\left|h'(t-s)\right| l(s)\varphi_3(y(s))\,ds + \left|h'(t-1)\right| \int_1^t \tau^{\alpha} l(\tau)\varphi_3(y(\tau))\,d\tau \\
&\quad + t^{\alpha} l(t)\varphi_3(y(t)) \int_1^t \left|h'(\sigma-1)\right| d\sigma - \int_1^t \sigma^{\alpha}\left|h'(t-\sigma)\right| l(\sigma)\varphi_3(y(\sigma))\,d\sigma \\
&= t^{\alpha}h_1(t)\varphi_1(w(t)) + h_2(t)\varphi_2(w(t)) + t^{\alpha}h(0)l(t)\varphi_3(w(t)) \\
&\quad + \left|h'(t-1)\right| \int_1^t \tau^{\alpha} l(\tau)\varphi_3(w(\tau))\,d\tau + t^{\alpha} l(t)\varphi_3(w(t)) \int_1^t \left|h'(\tau-1)\right| d\tau.
\end{aligned}$$

Integrating both sides of this inequality over $[1,t]$ gives

$$\begin{aligned}
w(t) &\leq c + \int_1^t s^{\alpha}h_1(s)\varphi_1(w(s))ds + \int_1^t h_2(s)\varphi_2(w(s))ds \\
&\quad + h(0)\int_1^t s^{\alpha} l(s)\varphi_3(w(s))\,ds + \int_1^t \left|h'(s-1)\right| \left(\int_1^s \tau^{\alpha} l(\tau)\varphi_3(w(\tau))\,d\tau\right) ds \\
&\quad + \int_1^t s^{\alpha} l(s)\varphi_3(w(s)) \left(\int_1^s \left|h'(\tau-1)\right| d\tau\right) ds \\
&= c + \int_1^t s^{\alpha}h_1(s)\varphi_1(w(s))ds + \int_1^t h_2(s)\varphi_2(w(s))ds \\
&\quad + h(0)\int_1^t s^{\alpha} l(s)\varphi_3(w(s))\,ds + \int_1^t \tau^{\alpha} l(\tau) \left(\int_{\tau}^t \left|h'(s-1)\right| ds\right) \varphi_3(w(\tau))\,d\tau \\
&\quad + \int_1^t s^{\alpha} l(s) \left(\int_1^s \left|h'(\tau-1)\right| d\tau\right) \varphi_3(w(s))\,ds \\
&= c + \int_1^t s^{\alpha}h_1(s)\varphi_1(w(s))ds + \int_1^t h_2(s)\varphi_2(w(s))ds + \int_1^t H(s)\varphi_3(w(s))\,ds,
\end{aligned}$$

where

$$H(s) = h(0)s^{\alpha} l(s) + s^{\alpha} l(s) \int_s^{\infty} \left|h'(\tau-1)\right| d\tau + s^{\alpha} l(s) \int_1^s \left|h'(\tau-1)\right| d\tau.$$

Lemma 3.23 allows us to conclude the result. ▮

Here is our main result of this subsection.

Theorem 4.4 *Suppose that the functions f and k satisfy ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_6$). Then, any solution of the problem* $(4.12)-(4.13)$ *is asymptotic to bt^α when $t \to \infty$, for some $b \in \mathbb{R}$.*

Proof. Under the condition ($\mathbf{H}_6$), the inequality, (4.42) implies that there exists a constant $B_5 > 0$ such that

$$u(t) \leq B_5 \text{ for all } \ t \geq 1. \tag{4.44}$$

Thus,

$$\frac{|x(t)|}{t^\alpha} \leq B_5, \quad \text{for all } \ t \geq 1, \tag{4.45}$$

and

$$\begin{aligned}
&\int_0^t \left| f\left(s, x(s), \int_0^s k\left(s, \tau, x(\tau)\right) d\tau\right) \right| ds \\
&\leq \int_0^1 \left(h_1(s) g_1(|x(s)|) + h_2(s) g_2 \left(\int_0^s h(s-\tau) g_3(|x(\tau)|) d\tau \right) \right) ds \\
&+ \int_1^t \left(s^\alpha h_1(s) \varphi_1 \left(\frac{|x(s)|}{s^\alpha} \right) + h_2(s) \varphi_2\left(u\left(s\right)\right) \right) ds.
\end{aligned}$$

The first integral on the right-hand side is finite by the continuity of the integrand functions over $[0,1]$ and the second one is uniformly bounded by (4.44) and (4.45). The conclusion is achieved as in the proof of Theorem 4.2. ▮

Example 4.2 *The functions*

$$k\left(t,s,x(s)\right)=\frac{e^{-t}}{1+te^{s}}\left(|x(s)|\right)^{\lambda},\ h\left(t\right)=e^{-t},\ l\left(t\right)=e^{-2t}\ \text{and}\ g_{3}\left(t\right)=t^{\lambda},\ 0\leq\lambda\leq1,$$

all satisfy ($\boldsymbol{H}_6$)*. Indeed, it is obvious that*

$$\begin{aligned}
&\int_1^{\infty}t^{\alpha}l\left(t\right)dt<\int_0^{\infty}t^{\alpha}e^{-t}dt=\Gamma(\alpha+1)<\infty,\\
&\int_1^{\infty}\left|h'(t-1)\right|dt=\int_1^{\infty}e^{-(t-1)}dt<\infty,\\
&\int_1^{t}s^{\alpha}h'(t-s)l\left(s\right)ds=-\int_1^{t}s^{\alpha}e^{-(t-s)}e^{-2s}ds=-e^{-t}\int_1^{t}s^{\alpha}e^{-s}ds,\\
&\int_1^{\infty}\left|\int_1^{t}s^{\alpha}h'(t-s)l\left(s\right)ds\right|dt=\int_1^{\infty}e^{-t}\left(\int_1^{t}s^{\alpha}e^{-s}ds\right)dt\\
&\leq\int_1^{\infty}e^{-t}\left(\int_1^{\infty}s^{\alpha}e^{-s}ds\right)dt\leq\Gamma(\alpha+1)\int_1^{\infty}e^{-t}dt<\infty.
\end{aligned}$$

4.4.4 Case of a Singular Kernel

Notice that, in all the above subsections, we assumed the continuity of the kernel k on $[0,\infty)$. Here we treat the singular kernel appeared in the fractional integral operator I_{0+}^{β}.

Consider the following equation

$$\left({}^{C}D_{0+}^{\alpha}x\right)'(t)=f\left(t,x(t),\ \left(I_{0+}^{\beta}x\right)(t)\right),\ t\geq0,\ 0<\alpha<1,\ \beta>0,\tag{4.46}$$

subject to

$$x(0)=c_0,\quad\left({}^{C}D_{0+}^{\alpha}x\right)(0)=c_1,\quad c_0,c_1\in\mathbb{R}.\tag{4.47}$$

We suppose that the function f satisfies the following hypotheses

($\mathbf{H}_7$) f satisfies ($\mathbf{H}_1$) and there are functions $h_1 \in \mathcal{H}_\alpha, h_2 \in \mathcal{H}_{\alpha+\beta}$ and $g_i \in \mathcal{G}_2$, $i = 1, 2$ with $g_1 \propto g_2$ such that

$$|f(t,u,v)| \leq h_1(t)g_1(|u|) + h_2(t)g_2(|v|), \;\; (t,u,v) \in D.$$

Lemma 4.11 *If x is a solution for the problem* (4.46) – (4.47) *and the functions f satisfies ($\boldsymbol{H}_7$), then*

$$\frac{|x(t)|}{t^\alpha}, \; \frac{\left|\left(I_{0^+}^\beta x\right)(t)\right|}{t^{\alpha+\beta}} \leq z(t), \quad \text{for all} \;\; t \geq 1, \tag{4.48}$$

where

$$\begin{aligned} z(t) \;\; &= \;\; A_1 + A_2 \int_0^t h_1(s) g_1(|x(s)|) ds \\ &\quad + A_2 \int_0^t h_2(s) g_2\left(\left|\left(I_{0^+}^\beta x\right)(s)\right|\right) ds, \;\; t \geq 0, \end{aligned} \tag{4.49}$$

$$\begin{aligned} A_1 \;\; &= \;\; \max\left\{|c_0| + \frac{|c_1|}{\Gamma(\alpha+1)}, \frac{|c_0|}{\Gamma(\beta+1)} + \frac{|c_1|}{\Gamma(\alpha+\beta+1)}\right\}, \\ A_2 \;\; &= \;\; \left\{\frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(\alpha+\beta+1)}\right\}. \end{aligned}$$

Proof. Integrating both sides of (4.46) over $[0,t]$ yields

$$\left({}^C D_{0^+}^\alpha x\right)(t) = c_1 + \int_0^t f\left(s, x(s), \left(I_{0^+}^\beta x\right)(s)\right) ds. \tag{4.50}$$

Next, applying $I^{\alpha}_{a^+}$ to both sides of equation (4.50) gives

$$x(t) = c_0 + \frac{c_1 t^\alpha}{\Gamma(\alpha+1)} + \left(I^{\alpha+1}_{0^+} f\left(s, x(s), \left(I^{\beta}_{0^+}x\right)(s)\right)\right)(t),\ t \geq 0,$$

and

$$|x(t)| \leq |c_0| + \frac{|c_1|\,t^\alpha}{\Gamma(\alpha+1)} + \frac{t^\alpha}{\Gamma(\alpha+1)} \int_0^t \left| f\left(s, x(s), \left(I^{\beta}_{0^+}x\right)(s)\right)\right| ds,\ t \geq 0. \quad (4.51)$$

For $t \geq 1$, (4.51) implies that

$$\frac{|x(t)|}{t^\alpha} \leq |c_0| + \frac{|c_1|}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \int_0^t \left| f\left(s, x(s), \left(I^{\beta}_{0^+}x\right)(s)\right)\right| ds.$$

From condition $(\mathbf{H}_7)$, we have

$$\begin{aligned} \frac{|x(t)|}{t^\alpha} &\leq |c_0| + \frac{|c_1|}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \int_0^t h_1(s) g_1(|x(s)|) ds \\ &\quad + \frac{1}{\Gamma(\alpha+1)} \int_0^t h_2(s) g_2\left(\left|\left(I^{\beta}_{0^+}x\right)(s)\right|\right) ds\ ,\ \text{for all } t \geq 1, \quad (4.52) \end{aligned}$$

Also, for all $t \geq 0$,

$$\begin{aligned} \left(I^{\beta}_{0^+}x\right)(t) &= I^{\beta}_{0^+}\left(c_0 + \frac{c_1 s^\alpha}{\Gamma(\alpha+1)} + \left(I^{\alpha+1}_{0^+} f\left(\tau, x(\tau), \left(I^{\beta}_{0^+}x\right)(\tau)\right)\right)(s)\right)(t) \\ &= \frac{c_0 t^\beta}{\Gamma(\beta+1)} + \frac{c_1 t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \left(I^{\alpha+\beta+1}_{0^+} f\left(\tau, x(\tau), \left(I^{\beta}_{0^+}x\right)(\tau)\right)\right)(t), \end{aligned}$$

$$\left|\left(I^{\beta}_{0^+}x\right)(t)\right| \leq \frac{|c_0|\,t^\beta}{\Gamma(\beta+1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(|c_1| + \int_0^t f\left|\left(s, x(s), \left(I^{\beta}_{0^+}x\right)(s)\right)\right| ds\right).$$

For $t \geq 1$, we deduce that

$$\begin{aligned}
\frac{\left|\left(I_{0^+}^{\beta} x\right)(t)\right|}{t^{\alpha+\beta}} \leq\ & \frac{|c_0|}{\Gamma(\beta+1)} + \frac{|c_1|}{\Gamma(\alpha+\beta+1)} + \frac{1}{\Gamma(\alpha+\beta+1)} \int_0^t h_1(s) g_1(|x(s)|) ds \\
& + \frac{1}{\Gamma(\alpha+\beta+1)} \int_0^t h_2(s) g_2\left(\left|\left(I_{0^+}^{\beta} x\right)(s)\right|\right) ds\ , \qquad (4.53)
\end{aligned}$$

for all $t \geq 1$. The result is obtained from (4.52) and (4.53). ∎

The next theorem represents the main result of this subsection.

Theorem 4.5 *Suppose that the functions f satisfies ($\boldsymbol{H}_7$). Then, any solution of the problem* (4.46) $-$ (4.47) *is asymptotic to bt^{α} when $t \to \infty$, for some $b \in \mathbb{R}$.*

Proof. Differentiating the expression z, given by (4.49), we obtain

$$z'(t) = A_2 h_1(t) g_1(|x(t)|) + A_2 h_2(t) g_2\left(\left|\left(I_{0^+}^{\beta} x\right)(t)\right|\right),\ t \geq 0.$$

We concluded from (4.48) and the monotonicity of g_1 and g_2, that

$$z'(t) \leq A_2 t^{\alpha} h_1(t) g_1(z(t)) + A_2 t^{\alpha+\beta} h_2(t) g_2(z(t)). \qquad (4.54)$$

Integrating both sides of (4.54) over $[1, t]$ leads to

$$z(t) \leq z(1) + A_2 \int_1^t s^{\alpha} h_1(s) g_1(z(s)) ds + A_2 \int_1^t s^{\alpha+\beta} h_2(s) g_2(z(s)) ds. \qquad (4.55)$$

As $h_1 \in \mathcal{H}_{\alpha}, h_2 \in \mathcal{H}_{\alpha+\beta}$ and $g_i \in \mathcal{G}_2$, applying Lemma 3.23 to (4.55) gives, for all

$t \geq 1$,

$$z(t) \leq A_3, \;\; A_3 > 0 \text{ is a constant.} \tag{4.56}$$

Recalling (4.49) and (4.50), we deduce

$$\frac{\left| \left({}^C D_{0^+}^{\alpha} x \right)(t) \right|}{\Gamma(\alpha+1)} \leq z(t) \text{ for all } \; t \geq 0.$$

In view of the relation (4.48), it follows that

$$\frac{\left| \left({}^C D_{0^+}^{\alpha} x \right)(t) \right|}{\Gamma(\alpha+1)}, \;\; \frac{|x(t)|}{t^{\alpha}}, \;\; \frac{\left| \left(I_{0^+}^{\beta} x \right)(t) \right|}{t^{\alpha+\beta}} \leq A_3, \quad \text{for all } \; t \geq 1. \tag{4.57}$$

Now, the proof can be completed by following the same process of the above proofs. ▌

4.4.5 Case of Fractional Source Terms

Now, we are going to generalize the result of Theorem 4.1 to the problem (4.5) – (4.6), where $0 \leq \beta \leq \alpha < 1$ and $0 \leq \gamma \leq \alpha < 1$. We start by estimating the two derivatives ${}^C D_{0^+}^{\beta} x$ and ${}^C D_{0^+}^{\gamma} x$ as shown in the following lemma.

Lemma 4.12 *Let x be a solution of problem* $(4.5)-(4.6)$. *Then,*

$$\begin{aligned}\left|({}^CD^{\beta}_{0^+}x)(t)\right| &\leq \frac{|c_1|\,t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\ &\quad\times\int_0^t\left|f\left(s,({}^CD^{\beta}_{0^+}x)\,(s)\,,\int_0^s k(s,\tau,\left({}^CD^{\gamma}_{0^+}x\right)(\tau))d\tau\right)\right|ds,\\ \left|\left({}^CD^{\gamma}_{0^+}x\right)(t)\right| &\leq \frac{|c_1|\,t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}+\frac{t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\\ &\quad\times\int_0^t\left|f\left(s,({}^CD^{\beta}_{0^+}x)\,(s)\,,\int_0^s k(s,\tau,\left({}^CD^{\gamma}_{0^+}x\right)(\tau))d\tau\right)\right|ds,\end{aligned}$$

for all $t>0$.

Proof. We know that applying $I^{\alpha+1}_{a^+}$ to both sides of equation (4.8) yields

$$x(t)=c_0+\frac{c_1t^{\alpha}}{\Gamma(\alpha+1)}+\left(I^{\alpha+1}_{0^+}f\left(s,({}^CD^{\beta}_{0^+}x)(s),\int_0^s k(s,\tau,\left({}^CD^{\gamma}_{0^+}x\right)(\tau))d\tau\right)\right)(t). \tag{4.58}$$

With help of Lemma 3.5 and Lemma 3.13, ${}^CD^{\beta}_{0^+}x$ and ${}^CD^{\gamma}_{0^+}x$ take the forms

$$\begin{aligned}({}^CD^{\beta}_{0^+}x)(t) &= \frac{c_1t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\ &\quad+\left(I^{\alpha-\beta+1}_{0^+}f\left(s,({}^CD^{\beta}_{0^+}x)\,(s)\,,\int_0^s k\left(s,\tau,\left({}^CD^{\gamma}_{0^+}x\right)(\tau)\right)d\tau\right)\right)(t)\,,\end{aligned}$$

$$\begin{aligned}\left({}^CD^{\gamma}_{0^+}x\right)(t) &= \frac{|c_1|\,t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\\ &\quad+\left(I^{\alpha-\gamma+1}_{0^+}f\left(s,({}^CD^{\beta}_{0^+}x)\,(s)\,,\int_0^s k\left(s,\tau,\left({}^CD^{\gamma}_{0^+}x\right)(\tau)\right)d\tau\right)\right)(t)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|({}^{C}D_{0^+}^{\beta}x)(t)\right| &\leq \frac{|c_1|\, t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\quad + \left(I_{0^+}^{\alpha-\beta+1}\left|f\left(s, ({}^{C}D_{0^+}^{\beta}x)(s), \int_0^s k\left(s,\tau,\left({}^{C}D_{0^+}^{\gamma}x\right)(\tau)\right) d\tau\right)\right|\right)(t) \\
&= \frac{|c_1|\, t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} \\
&\quad \times \int_0^t (t-s)^{\alpha-\beta}\left|f\left(s, ({}^{C}D_{0^+}^{\beta}x)(s), \int_0^s k(s,\tau,\left({}^{C}D_{0^+}^{\gamma}x\right)(\tau))d\tau\right)\right| ds \\
&\leq \frac{|c_1|\, t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\quad \times \int_0^t \left|f\left(s, ({}^{C}D_{0^+}^{\beta}x)(s), \int_0^s k(s,\tau,\left({}^{C}D_{0^+}^{\gamma}x\right)(\tau))d\tau\right)\right| ds,
\end{aligned}$$

for all $t > 0$.

The same can be done for $\left|\left({}^{C}D_{0^+}^{\gamma}x\right)(t)\right|$. ∎

Lemma 4.6 can be generalized without major changes in the proof.

Lemma 4.13 *Let*

$$\begin{aligned}
u_1(t) &= A_1 + A_2\int_0^t h_1(s)g_1(\left|({}^{C}D_{0^+}^{\beta}x)(s)\right|)ds, \\
u_2(t) &= A_2\int_0^t h_2(s)g_2\left(u_3(s)\right)ds, \\
u_3(t) &= \int_0^t h_3(s)g_3(\left|\left({}^{C}D_{0^+}^{\gamma}x\right)(s)\right|)ds,
\end{aligned}$$

for all $t \geq 0$, where $h_1 \in \mathcal{H}_{\alpha-\beta}$, $h_2 \in \mathcal{H}_0$, $h_3 \in \mathcal{H}_{\alpha-\gamma}$ and g_i are of type $\mathcal{G}_2$,

$i=1,2,3$. *Assume that x is a solution for the problem* $(4.5)-(4.6)$. *Then,*

$$\begin{aligned} u(t) &\leq u(1)+A_2\int_1^t s^{\alpha-\beta}h_1(s)g_1(u(s))ds+A_2\int_1^t h_2(s)g_2(u(s))ds \\ &\quad +\int_1^t s^{\alpha-\gamma}h_3(s)\,g_3(u(s))ds,\ \textit{for all } t\geq 1, \qquad (4.59)\end{aligned}$$

where

$$u=u_1+u_2+u_3, \qquad (4.60)$$

and

$$\begin{aligned} u(1) &= A_1+A_2\int_0^1 h_1(s)g_1(\left|({}^{C}D_{0^+}^{\beta}x)(s)\right|)ds+A_2\int_0^1 h_2(s)g_2(u_3(s))\,ds \\ &\quad +\int_0^1 h_3(s)g_3\left(\left|\left({}^{C}D_{0^+}^{\gamma}x\right)(s)\right|\right)ds. \qquad (4.61)\end{aligned}$$

Theorem 4.6 *Suppose that the functions f and k satisfy ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_3$) with $h_1\in\mathcal{H}_{\alpha-\beta}$, $h_2\in\ \mathcal{H}_0$, $h_3\in\mathcal{H}_{\alpha-\gamma}$. Then, any solution of the problem* $(4.5)-(4.6)$ *is asymptotic to bt^{α} when $t\to\infty$, for some $b\in\mathbb{R}$.*

Proof. We first note that the relation (4.59), implies, in view of Lemma 3.23, that

$$u(t)\leq G_3^{-1}\left(G_3(C_2)+\int_1^t s^{\alpha-\gamma}h_3(s)ds\right),\ t\geq 1,$$

where $C_2=G_2^{-1}\left(G_2(C_1)+\int_1^{\infty}h_2(s)ds\right)$, $C_1=G_1^{-1}\left(G_1(C_0)+\int_1^{\infty}s^{\alpha-\beta}h_1(s)ds\right)$, $C_0=A_3$. Therefore, there exists a constant $A_7>0$ such that

$$\frac{\left|{}^{C}D_{0^+}^{\beta}x(t)\right|}{t^{\alpha-\beta}}\leq A_7\ \text{ for all }\ t\geq 1, \qquad (4.62)$$

and consequently by exploiting the continuity of ${}^{C}D_{0+}^{\beta}x$, ${}^{C}D_{0+}^{\gamma}x$ (see Lemma 3.13), we see that

$$
\begin{aligned}
&\int_0^t \left| f\left(s, ({}^{C}D_{0+}^{\beta}x)(s), \int_0^s k\left(s,\tau,\left({}^{C}D_{0+}^{\gamma}x\right)(\tau)\right)d\tau\right)\right| ds \\
&\leq \int_0^1 \left(h_1(s)g_1(\left|({}^{C}D_{0+}^{\beta}x)(s)\right|) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(\left|\left({}^{C}D_{0+}^{\gamma}x\right)(\tau)\right|)d\tau\right)\right) ds \\
&+ \int_1^t \left(s^{\alpha-\beta}h_1(s)g_1\left(\frac{\left|({}^{C}D_{0+}^{\beta}x)(s)\right|}{s^{\alpha-\beta}}\right) + h_2(s)g_2\left(u\left(s\right)\right)\right) ds,
\end{aligned}
$$

is uniformly bounded for all $t > 0$, and as in the proof of previous theorems

$$\lim_{t\to\infty} \frac{\left({}^{C}D_{0+}^{\alpha}x\right)(t)}{\Gamma(\alpha+1)} = b = \lim_{t\to\infty}\frac{x(t)}{t^{\alpha}}.$$

The proof is complete. ∎

4.5 Equations with Riemann-Liouville Fractional Derivatives

When $1 < \mu < 2$, we write $D_{0+}^{\mu} = D_{0+}^{\alpha+1}$, $0 < \alpha < 1$, in the equation (1.1). It is clear that $DD_{0+}^{\alpha} = \left(D_{0+}^{\alpha}\right)' = D_{0+}^{\alpha+1}$ in case of the Riemann-Liouville fractional derivative.

In this section, we consider the equation

$$\left(D_{0+}^{\alpha+1}x\right)(t) = f\left(t, (D_{0+}^{\beta}x)(t), \int_0^t k\left(t,s,\left(D_{0+}^{\gamma}x\right)(s)\right)ds\right),\ t>0, \tag{4.63}$$

subject to

$$\left(I_{0^+}^{1-\alpha}x\right)(0^+) = a_1, \;\; \left(\; D_{0^+}^{\alpha}x\right)(0^+) = a_2, \quad a_1, a_2 \in \mathbb{R}, \tag{4.64}$$

where $D_{0^+}^{\alpha+1}, D_{0^+}^{\beta}$ and $D_{0^+}^{\gamma}$ are the Riemann-Liouville fractional derivatives of orders $\alpha+1$, β and γ, respectively, $0 \leq \beta \leq \alpha < 1$ and $0 \leq \gamma \leq \alpha < 1$. We assume that the functions f and k satisfy the hypotheses

($\tilde{\mathbf{H}}_1$) $f(t,u,v)$ is a $C_{1-\alpha}$ function in $D = \{(t,u,v) : t \geq 0,\; u,\; v \in \mathbb{R}\}$.

and ($\mathbf{H}_2$), respectively.

We study the asymptotic behavior of solutions for the problem (4.63) – (4.64) in the sense of the following definition.

Definition 4.8 *We mean by a solution x of* (4.63)$-$(4.64), *a function $x : (0,b] \to \mathbb{R}$, that it is continuable, satisfies the equation* (4.63) *and the initial conditions* (4.64) *and is in the space $C_{1-\alpha}^{\alpha+1}[0,b]$, $0 < b \leq \infty$, defined by*

$$C_{1-\alpha}^{\alpha+1}[0,b] = \left\{x : (0,b] \to \mathbb{R} \mid x \in C_{1-\alpha}[0,b],\; D_{0^+}^{\alpha+1}x \in C_{1-\alpha}[0,b]\right\}, \tag{4.65}$$

where the space $C_{1-\alpha}[0,b]$ is defined in (3.1).

4.5.1 Some Useful Lemmas

The following lemmas will be needed in the next subsections.

Lemma 4.14 *Let $0 < \alpha < 1$. If $x \in C_{1-\alpha}^{\alpha+1}[0,b]$, then $D_{0^+}^{\alpha}x \in C[0,b]$.*

Proof. Note that if $x \in C_{1-\alpha}^{\alpha+1}[0,b]$, then $D_{0^+}^{\alpha+1}x = D_{0^+}^{2} I_{0^+}^{1-\alpha}x \in C_{1-\alpha}[0,b]$ and we have from Lemma 3.2, $I_{0^+}^{1-\alpha}x \in C_{1-\alpha}^{2}[0,b] \subset AC^2[0,b] \subset C^1[0,b]$. Therefore $DI_{0^+}^{1-\alpha}x = D_{0^+}^{\alpha}x \in C[0,b]$. ∎

Lemma 4.15 *Let x be a solution of problem* (4.63) – (4.64) *with $f \in L^1(0,\infty)$. Then,*

$$\begin{aligned}
\lim_{t\to\infty} \frac{x(t)}{t^\alpha} &= \lim_{t\to\infty} \frac{\left(D_{0^+}^{\alpha}x\right)(t)}{\Gamma(\alpha+1)} \\
&= \frac{1}{\Gamma(\alpha+1)} \left(a_2 + \int_0^\infty f\left(s, \left(D_{0^+}^{\beta}x\right)(s), \int_0^s k(s,\tau,\left(D_{0^+}^{\gamma}x\right)(\tau))d\tau \right) ds \right).
\end{aligned}$$

Proof. Applying $I_{0^+}^{1}$ to both sides of equation (4.63) and then applying $I_{a^+}^{\alpha}$ to the resulting equation, taking into account Lemmas (3.10), 3.16 and 3.5, we get

$$\left(D_{0^+}^{\alpha}x\right)(t) = a_2 + \int_0^t f\left(s, (D_{0^+}^{\beta}x)(s), \int_0^s k\left(s,\tau,\left(D_{0^+}^{\gamma}x\right)(\tau)\right)d\tau\right)ds, \tag{4.66}$$

$$x(t) = \frac{a_1 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{a_2 t^\alpha}{\Gamma(\alpha+1)} + \left(I_{0^+}^{\alpha+1} f\left(s, (D_{0^+}^{\beta}x)(s), \int_0^s k(s,\tau,\left(D_{0^+}^{\gamma}x\right)(\tau))d\tau\right)\right)(t). \tag{4.67}$$

for all $t > 0$. Taking the limit of the ratio $\frac{x(t)}{t^\alpha}$ as $t \to \infty$ gives the desired result with help of Lemma 4.3. ∎

The next two lemmas provide estimates for some integrals which will appear later in our results.

Lemma 4.16 *Let b_2, b_3 and b_4 be positive constants and $z(t)$ be a continuous and* *nonnegative function on* $[0,\infty)$. *Assume that*

$$z(t) \le b_2 + b_3 t + b_4 t \int_0^t \left(h_1(s) g_1\left(z(s)\right) + h_2(s) g_2\left(z(s)\right)\right) ds, \ t \ge 0, \qquad (4.68)$$

where $h_1, h_2 \in \mathcal{H}_1$ *and* $g_1, g_2 \in \mathcal{G}_2$ *with* $g_1 \propto g_2$. *Then,*

$$z(t) \le \begin{cases} G_2^{-1}\left(d_2\right), \ 0 \le t < 1 \\ tG_2^{-1}\left(d_3\right), \ t \ge 1 \end{cases},$$

where

$$\begin{aligned} d_2 &= G_2\left(d_1\right) + \int_0^1 h_2(s) ds, \ d_1 = G_1^{-1}\left(G_1(d_0) + \frac{\int_0^1 h_1(s) ds}{\Gamma(\alpha+1)}\right), \ d_0 = b_2 + b_3, \\ d_3 &= G_2(e_1) + b_4 \int_1^\infty s h_2(s) ds, \ e_1 = G_1^{-1}\left(G_1(d_4) + b_4 \int_1^\infty s h_1(s) ds\right), \\ d_4 &= d_0 + b_4 g_1(G_2^{-1}\left(d_2\right)) \int_0^1 h_1(s) ds + b_4 g_2\left(G_2^{-1}\left(d_2\right)\right) \int_0^1 h_2(s) ds, \end{aligned}$$

and G_i^{-1} *is the inverse function of* $G_i\left(t\right) = \int_{t_0}^t \frac{d\tau}{g_i(\tau)}$, $i = 1, 2$, $t_0 > 0$, $t > 0$.

Proof. For $0 \le t < 1$ we obtain from (4.68)

$$z(t) \le b_2 + b_3 + b_4 \int_0^t h_1(s) g_1(z(s)) ds + b_4 \int_0^t h_2(s) g_2\left(z(s)\right) ds.$$

It follows from Lemma 3.23 that

$$z(t) \le G_2^{-1}\left(d_2\right).$$

For $t \geq 1$, we have from (4.68)

$$\begin{aligned}
\frac{z(t)}{t} &\leq b_2 + b_3 + b_4 \int_0^t h_1(s) g_1(z(s)) ds + b_4 \int_0^t h_2(s) g_2 \left(z(s) \right) ds \\
&\leq d_4 + b_4 \int_1^t s h_1(s) g_1 \left(\frac{z(s)}{s} \right) ds + b_4 \int_1^t s h_2(s) g_2 \left(\frac{z(s)}{s} \right) ds. \quad (4.69)
\end{aligned}$$

Now, the hypotheses of Lemma 3.23 are satisfied with $b_1 = \infty$ (because $\int_{t_0}^{\infty} \frac{d\tau}{g_i(\tau)} = \infty$, $i = 1, 2$). Then, (4.69) for $t \in [1, \infty)$ leads to

$$\frac{z(t)}{t} \leq G_2^{-1} \left(d_3 \right).$$

This completes the proof. ∎

Lemma 4.17 *Let b_2, b_3 and b_4 be positive constants and let $z(t)$ be a continuous and nonnegative function on $[0, \infty)$. If*

$$z(t) \leq b_2 + b_3 t + b_4 t \int_0^t \left(h_1(s) g_1(z(s)) + h_2(s) g_2 \left(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau \right) \right) ds, \; t \geq 0, \quad (4.70)$$

where h_1, h_3 are of type $\mathcal{H}_1$, h_2 is of type $\mathcal{H}_0$ and g_i is of type $\mathcal{G}_2$, $i = 1, 2, 3$ with $g_1 \propto g_2 \propto g_3$. Then,

$$z(t) \leq \begin{cases} G_3^{-1} \left(M \right), \; 0 \leq t < 1 \\ t G_3^{-1} \left(M_1 \right), \; t \geq 1 \end{cases},$$

where

$$\begin{aligned}
M &= G_3\left(d_2\right) + \int_0^1 h_3(s)ds,\ d_2 = G_2^{-1}\left(G_2(d_1) + b_4\int_0^1 h_2(s)ds\right), \\
d_1 &= G_1^{-1}\left(G_1(d_0) + b_4\int_0^1 h_1(s)ds\right),\ d_0 = b_2 + b_3, \\
M_1 &= G_3(e_2) + \int_1^\infty sh_3(s)ds,\ e_2 = G_2^{-1}\left(G_2(e_1) + b_4\int_1^\infty h_2(s)ds\right), \\
e_1 &= G_1^{-1}\left(G_1(M_2) + b_4\int_1^\infty sh_1(s)ds\right), \\
M_2 &= d_0 + b_4 g_1(G_3^{-1}(M))\int_0^1 h_1(s)ds + b_4 g_2\left(g_3\left(G_3^{-1}(M)\right)\int_0^1 h_3(\tau)d\tau\right)\int_0^1 h_2(s)ds.
\end{aligned}$$

Proof. For $0 \le t < 1$, we obtain from (4.70)

$$z(t) \le b_2 + b_3 + b_4\int_0^t \left(h_1(s)g_1\left(z(s)\right) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(z(\tau))d\tau\right)\right)ds.$$

It follows from Lemma 3.24 that

$$z(t) \le G_3^{-1}(M) \quad \text{for all } 0 \le t < 1.$$

For $t \ge 1$, we have from (4.70),

$$\begin{aligned}
\frac{z(t)}{t} &\le d_0 + b_4\int_0^1 \left(h_1(s)g_1(G_3^{-1}(M)) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(G_3^{-1}(M))d\tau\right)\right)ds \\
&\quad + b_4\int_1^t \left(h_1(s)g_1(z(s)) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(z(\tau))d\tau\right)\right)ds \\
&\le M_2 + b_4\int_1^t \left(h_1(s)g_1(z(s)) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(z(\tau))d\tau\right)\right)ds.
\end{aligned}$$

Let $u = u_1 + u_2 + u_3$, where

$$\begin{aligned} u_1(t) &= M_2 + b_4 \int_0^t h_1(s) g_1(z(s)) ds, \\ u_2(t) &= b_4 \int_0^t h_2(s) g_2(u_3(s))\, ds, \quad u_3(t) = \int_0^t h_3(s) g_3(z(s)) ds, \ t > 0. \end{aligned}$$

Differentiating u, we get, in virtue of the monotonicity of $g_i, i = 1, 2, 3$,

$$u'(t) \leq b_4 t h_1(t) g_1(u(t)) + b_4 h_2(t) g_2(u(t)) + t h_3(t) g_3(u(t)). \tag{4.71}$$

for all $t \geq 1$. Integrating both sides of (4.71) over $[1, t]$ gives

$$\begin{aligned} u(t) \leq\ & u(1) + b_4 \int_1^t s h_1(s) g_1(u(s)) ds + b_4 \int_1^t h_2(s) g_2(u(s)) ds \\ & + \int_1^t s h_3(s) g_3(u(s)) ds. \end{aligned} \tag{4.72}$$

Now, since $\int_{t_0}^{\infty} \frac{d\tau}{g_i(\tau)} = \infty$, $i = 1, 2, 3$ for any $t_0 > 0$, the hypotheses of Lemma 3.23 are satisfied with $b_1 = \infty$. Therefore, by Lemma 3.23, the inequality (4.72) leads to

$$u(t) \leq G_3^{-1}(M_1), \text{ for all } t \geq 1.$$

The proof is now complete. ▮

Although the estimates in Lemmas 4.16 and 4.17 are not the best, they ensure that all the involved integrals are bounded, which is the most useful fact we need in the next subsections.

4.5.2 Case of a Non-fractional Source

In this subsection, we consider the problem $(4.63)-(4.64)$ with $\beta=\gamma=0$ and $0<\alpha<1$, that is,

$$\left(D_{0^+}^{\alpha+1}x\right)(t)=f\left(t,x(t),\int_0^t k\left(t,s,x(s)\right)ds\right),\ t>0, \tag{4.73}$$

subject to

$$\left(I_{0^+}^{1-\alpha}x\right)(0^+)=a_1,\ \ \left(D_{0^+}^{\alpha}x\right)(0^+)=a_2,\ \ a_1,a_2\in\mathbb{R}. \tag{4.74}$$

First, we need the following condition:

($\tilde{\mathbf{H}}_3$) There are functions $h_1,h_3\in\mathcal{H}_1,h_2\in\mathcal{H}_0$ and $g_i\in\mathcal{G}_2,\ i=1,2,3$ with $g_1\propto g_2\propto g_3$ such that

$$|f(t,u,v)|\leq h_1(t)g_1\left(\frac{|u|}{t^{\alpha-1}}\right)+h_2(t)g_2(|v|),\ \ (t,u,v)\in D, \tag{4.75}$$

and

$$|k(t,s,x)|\leq h_3(s)g_3\left(\frac{|x|}{s^{\alpha-1}}\right),\ \ (t,s,x)\in E. \tag{4.76}$$

Now, we prove the main result in this subsection.

Theorem 4.7 *Suppose that f and k satisfy ($\tilde{\boldsymbol{H}}_1$), ($\boldsymbol{H}_2$) and ($\tilde{\boldsymbol{H}}_3$). Then, any solution of the problem* $(4.73)-(4.74)$ *is asymptotic to ct^{α} when $t\to\infty$, for some $c\in\mathbb{R}$.*

Proof. For the case of the Riemann-Liouville fractional derivative, applying

$I_{0^+}^{\alpha+1}$ to both sides of the equation in (4.73) yields

$$x(t) = \frac{a_1 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{a_2 t^{\alpha}}{\Gamma(\alpha+1)} + \left(I_{0^+}^{\alpha+1} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\right)\right)(t).$$

Then, for all $t > 0$,

$$\begin{aligned} \frac{|x(t)|}{t^{\alpha-1}} \leq\ & \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2|\, t}{\Gamma(\alpha+1)} + \frac{t}{\Gamma(\alpha+1)} \int_0^t \left[h_1(s) g_1\left(\frac{|x(s)|}{s^{\alpha-1}}\right) \right. \\ & \left. + h_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{|x(\tau)|}{\tau^{\alpha-1}}\right) d\tau \right) \right] ds. \end{aligned} \tag{4.77}$$

Let us denote the right hand side of inequality (4.77) by $z(t)$ for all $t > 0$, then

$$\frac{|x(t)|}{t^{\alpha-1}} \leq z(t), \text{ for all } t > 0, \tag{4.78}$$

and consequently,

$$\begin{aligned} z(t) \leq\ & \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2|}{\Gamma(\alpha+1)} t + \frac{t}{\Gamma(\alpha+1)} \int_0^t [h_1(s) g_1(z(s)) \\ & \left. + h_2(s) g_2 \left(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau \right) \right] ds \text{ for all } t > 0. \end{aligned} \tag{4.79}$$

It follows from Lemma 4.17 that

$$z(t) \leq t G_3^{-1}(M_1), \text{ for all } t \geq 1,$$

and we have from (4.78) that

$$\frac{|x(t)|}{t^\alpha} \le M_3 := G_3^{-1}(M_1), \text{ for all } t \ge 1. \tag{4.80}$$

Let

$$J := \int_0^t \left| f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\right)\right| ds, \ t > 0.$$

Using the assumption ($\tilde{\mathbf{H}}_3$) and (4.78) we see that

$$\begin{aligned} J \ &\le \ \int_0^1 \left[h_1(s)g_1(z(s)) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(z(\tau))d\tau\right)\right] ds \\ &\quad + \int_1^t \left[h_1(s)g_1(z(s)) + h_2(s)g_2\left(\int_0^s h_3(\tau)g_3(z(\tau))d\tau\right)\right] ds, \ t \ge 1. \end{aligned} \tag{4.81}$$

The second integral on the right-hand side of (4.81) can be estimated using (4.78) as follows

$$\begin{aligned} J_2 \ &\le \ \int_1^t sh_1(s)g_1(M_3)ds + \int_1^t h_2(s)g_2\left(\int_0^1 h_3(\tau)g_3\left(z(\tau)\right)d\tau\right. \\ &\quad \left. + \int_1^s h_3(\tau)g_3(z(\tau))d\tau\right) ds \\ &\le \ g_1(M_3)\int_1^t sh_1(s)ds + g_2\left(g_3(M_4)\int_0^1 h_3(\tau)d\tau + g_3(M_3)\int_1^t \tau h_3(\tau)d\tau\right) \\ &\quad \times \int_1^t h_2(s)ds. \end{aligned}$$

for all $t \ge 1$. As $h_1, h_3 \in \mathcal{H}_1, h_2 \in \mathcal{H}_0$, we deduce that J_2 is uniformly bounded and so is J.

It means that the integral $\int_0^t f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\right) ds$ is absolutely convergent and so

$$\lim_{t\to\infty}\int_0^t f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\right)ds < \infty. \tag{4.82}$$

Integrating both sides of (4.73) over the interval $[0, t]$ yields

$$\left(D_{0^+}^{\alpha}x\right)(t) = a_1 + \int_0^t f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\right)ds.$$

Now, (4.82) insures that there is a real number $\hat{c}$ such that

$$\lim_{t\to\infty} D_{0^+}^{\alpha}x(t) = \hat{c}.$$

By Lemma 4.15

$$\lim_{t\to\infty}\frac{x(t)}{t^{\alpha}} = \lim_{t\to\infty}\frac{\left(D_{0^+}^{\alpha}x\right)(t)}{\Gamma(\alpha+1)} = c,$$

$c := \frac{\hat{c}}{\Gamma(\alpha+1)}$. This completes the proof. ∎

4.5.3 Case of a Singular Kernel

Consider the following equation

$$D_{0^+}^{\alpha+1}x(t) = f\left(t, x(t), \left(I_{0^+}^{\beta}x\right)(t)\right), \ t>0, \ 0<\alpha<1, \ 0<\alpha+\beta<1, \tag{4.83}$$

subject to

$$\left(I_{0^+}^{1-\alpha}x\right)(0^+) = a_1, \ \left(D_{0^+}^{\alpha}x\right)(0^+) = a_2, \quad a_1, a_2 \in \mathbb{R}. \tag{4.84}$$

To study the asymptotic behavior of solutions for the problem $(4.83)-(4.84)$, we assume that the function f satisfies the condition

($\tilde{\mathbf{H}}_7$) There are functions $h_1, h_2 \in \mathcal{H}_1$ and $g_1, g_2 \in \mathcal{G}_2$ with $g_1 \propto g_2$ such that

$$|f(t,u,v)| \leq h_1(t)g_1\left(\frac{|u|}{t^{\alpha-1}}\right) + h_2(t)g_2\left(\frac{|v|}{t^{\alpha+\beta-1}}\right), \quad (t,u,v) \in D.$$

Theorem 4.8 *Suppose that f satisfies the conditions ($\tilde{\boldsymbol{H}}_1$), and ($\tilde{\boldsymbol{H}}_7$). Then, any solution of the problem $(4.83)-(4.84)$ is asymptotic to ct^{α} when $t \to \infty$, for some $c \in \mathbb{R}$.*

Proof. We have from the condition ($\tilde{\mathbf{H}}_7$) after applying $I_{0^+}^{\alpha+1}$ to both sides of equation (4.83),

$$\begin{aligned} t^{1-\alpha}|x(t)| \quad \leq \quad & \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2|\,t}{\Gamma(\alpha+1)} + \frac{t}{\Gamma(\alpha+1)}\int_0^t \left[h_1(s)g_1\left(\frac{|x(s)|}{s^{\alpha-1}}\right)\right. \\ & \left. + h_2(s)g_2\left(\frac{\left|\left(I_{0^+}^{\beta}x\right)(s)\right|}{s^{\alpha+\beta-1}}\right)\right] ds, \ t>0. \end{aligned} \tag{4.85}$$

As

$$\left(I_{0^+}^{\beta}x\right)(t) = \frac{a_1 t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{a_2 t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + I_{0^+}^{\alpha+\beta+1} f\left(\tau, x(\tau), \left(I_{0^+}^{\beta}x\right)(\tau)\right)(s)\,(t),$$

for all $t>0$, we arrive at

$$\begin{aligned}\left|\left(I_{0^+}^{\beta}x\right)(t)\right| &\leq \frac{|a_1|\,t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{|a_2|\,t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+I_{0^+}^{\alpha+\beta+1}\left|f\left(\tau,x(\tau),\left(I_{0^+}^{\beta}x\right)(\tau)\right)\right|(s)\,(t)\\ &\leq \frac{|a_1|\,t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(|a_2|+\int_0^t\left|f\left(s,x(s),\left(I_{0^+}^{\beta}x\right)(s)\right)\right|ds\right),\end{aligned}$$

or equivalently with help of $(\tilde{\mathbf{H}}_7)$,

$$\begin{aligned}t^{1-\alpha-\beta}\left|\left(I_{0^+}^{\beta}x\right)(t)\right| &\leq \frac{|a_1|}{\Gamma(\alpha+\beta)}+\frac{t}{\Gamma(\alpha+\beta+1)}\left(|a_2|+\int_0^t\left[h_1(s)g_1\left(\frac{|x(s)|}{s^{\alpha-1}}\right)\right.\right.\\ &\left.\left.+h_2(s)g_2\left(\frac{\left|\left(I_{0^+}^{\beta}x\right)(s)\right|}{s^{\alpha+\beta-1}}\right)\right]ds\right)\quad \text{for all } t>0. \qquad (4.86)\end{aligned}$$

Now, let

$$z(t)=A_1+A_2t+A_3t\int_0^t\left(h_1(s)g_1\left(\frac{|x(s)|}{s^{\alpha-1}}\right)+h_2(s)g_2\left(\frac{\left|\left(I_{0^+}^{\beta}x\right)(s)\right|}{s^{\alpha+\beta-1}}\right)\right)ds, \qquad (4.87)$$

for all $t>0$, where

$$\begin{aligned}A_1 &= \max\left\{\frac{|a_1|}{\Gamma(\alpha)},\frac{|a_1|}{\Gamma(\alpha+\beta)}\right\},\ A_2=\max\left\{\frac{|a_2|}{\Gamma(\alpha+1)},\frac{|a_2|}{\Gamma(\alpha+\beta+1)}\right\},\\ A_2 &= \max\left\{\frac{1}{\Gamma(\alpha+1)},\frac{1}{\Gamma(\alpha+\beta+1)}\right\}.\end{aligned}$$

It is not difficult to see from the relations(4.85) $-$ (4.87), that

$$t^{1-\alpha}\left|x(t)\right|,\ t^{1-\alpha-\beta}\left|\left(I_{0^+}^{\beta}x\right)(t)\right|\leq z(t),\quad t>0,$$

and consequently, for $t > 0$,

$$z(t) \leq A_1 + A_2 t + A_3 t \int_0^t h_1(s) g_1(z(s)) ds + A_3 t \int_0^t h_2(s) g_2 \left(z(s)\right) ds, \ t > 0.$$

It follows from Lemma 4.16 that

$$z(t) \leq t G_2^{-1} \left(d_3\right), \ \text{for all } t \geq 1,$$

where G_2^{-1} and d_3 are given in Lemma 4.16. Now, the proof can be completed in a similar manner of the proof of Theorem 4.7. ∎

4.5.4 Case of Delay

Here we consider the following equation

$$D_{0^+}^{\alpha+1} x(t) = f\left(t, x(\psi(t)), \int_0^t k(t, s, x(\psi(s))) ds\right), \ t > 0, \ 0 < \alpha < 1, \tag{4.88}$$

subject to

$$\left(I_{0^+}^{1-\alpha} x\right)\left(0^+\right) = a_1, \ \left(D_{0^+}^{\alpha} x\right)\left(0^+\right) = a_2, \quad a_1, a_2 \in \mathbb{R}. \tag{4.89}$$

In order to study the asymptotic of solutions for the problem $(4.88) - (4.89)$, we set the following condition:

(H$_8$) There are functions $h_1, h_3 \in \mathcal{H}_1, h_2 \in \mathcal{H}_0, \ g_i \in \mathcal{G}_2, \ i = 1, 2, 3$ with $g_1 \propto g_2 \propto g_3$ and a function ψ which is nonnegative continuous on $[0, \infty)$ with

$\psi(t) \leq t$ and $\lim\limits_{t\to\infty} \psi(t) = \infty$ such that

$$|f(t,u,v)| \leq h_1(t) g_1\left(\frac{|u|}{\psi^{\alpha-1}(t)}\right) + h_2(t) g_2(|v|),\ (t,u,v) \in D,$$

$$|k(t,s,x)| \leq h_3(s) g_3\left(\frac{|x|}{\psi^{\alpha-1}(s)}\right),\ (t,s,x) \in E.$$

Theorem 4.9 *Suppose that f and k satisfy the conditions ($\tilde{\boldsymbol{H}}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_8$). Then, any solution of the problem* $(4.88)-(4.89)$ *is asymptotic to ct^{α} when $t \to \infty$, for some $c \in \mathbb{R}$.*

Proof. We have here for all $t > 0$,

$$\frac{|x(t)|}{t^{\alpha-1}} \leq \frac{|a_1|}{\Gamma(\alpha)} + \frac{t}{\Gamma(\alpha+1)}\left(|a_2| + \left(I_0^1\left|f\left(s, x(\psi(s)), \int_0^s k(s,\tau,x(\psi(\tau)))d\tau\right)\right|\right)(t)\right).$$

Therefore,

$$\begin{aligned}\frac{|x(t)|}{t^{\alpha-1}} \quad &\leq \quad \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2|\, t}{\Gamma(\alpha+1)} + \frac{t}{\Gamma(\alpha+1)}\int_0^t \left[h_1(s) g_1\left(\frac{|x(\psi(s))|}{\psi^{\alpha-1}(s)}\right)\right.\\ &\quad \left. + h_2(s) g_2\left(\int_0^s h_3(\tau) g_3\left(\frac{|x(\psi(\tau))|}{\psi^{\alpha-1}(\tau)}\right) d\tau\right)\right] ds,\ t > 0. \qquad (4.90)\end{aligned}$$

Let us denote the right hand side of inequality (4.90) by $z(t)$ for all $t > 0$, then

$$\frac{|x(t)|}{t^{\alpha-1}} \leq z(t) \quad \text{and} \quad \frac{|x(\psi(t))|}{\psi^{\alpha-1}(t)} \leq z(\psi(t)),\ t > 0. \qquad (4.91)$$

We also see from the condition ($\mathbf{H}_8$) on ψ and from the monotonicity of z,

$$\frac{|x(\psi(t))|}{\psi^{\alpha-1}(t)} \leq z(t), \quad t > 0, \tag{4.92}$$

and thereafter,

$$\begin{aligned} z(t) \quad \leq \quad & \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2|}{\Gamma(\alpha+1)} t + \frac{t}{\Gamma(\alpha+1)} \int_0^t [h_1(s) g_1(z(s)) \\ & + h_2(s) g_2 \left(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau \right) \bigg] ds, \; t > 0. \end{aligned} \tag{4.93}$$

The rest of the proof is similar to that of Theorem 4.7. ∎

4.5.5 Case of Fractional Source Terms

We study in this subsection the asymptotic behavior of solutions for the problem (4.63) – (4.64) under the following condition:

($\mathbf{H}_{10}$) There are functions $h_1, h_3 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_0$ and $g_i \in \mathcal{G}_2, i = 1, 2, 3$, with $g_1 \propto g_2 \propto g_3$ such that

$$|f(t, u, v)| \leq h_1(t) g_1 \left(\frac{|u|}{t^{\alpha-\beta-1}} \right) + h_2(t) g_2(|v|), \;\; (t, u, v) \in D,$$

$$|k(t, s, x)| \leq h_3(s) g_3 \left(\frac{|x|}{t^{\alpha-\gamma-1}} \right), \;\; (t, s, x) \in E.$$

Now, we are ready to state and prove the main result of this subsection.

Theorem 4.10 *Suppose that f and k satisfy the conditions ($\tilde{\boldsymbol{H}}_1$), ($\boldsymbol{H}_2$) and*

($\boldsymbol{H}_{10}$). Then, any solution of the problem $(4.63)-(4.64)$ *is asymptotic to* ct^{α} *when* $t \to \infty$, *for some* $c \in \mathbb{R}$.

Proof. Here we have

$$x(t) = \frac{a_1 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{a_2 t^{\alpha}}{\Gamma(\alpha+1)} + \left(I_{0^+}^{\alpha+1} f\left(s, (D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau,\left(D_{0^+}^{\gamma} x\right)(\tau)) d\tau \right)\right)(t)\,, \tag{4.94}$$

$$\begin{aligned} \frac{|x(t)|}{t^{\alpha-1}} \quad \leq \quad & \frac{|a_1|}{\Gamma(\alpha)} + \frac{|a_2|\, t}{\Gamma(\alpha+1)} + \frac{t}{\Gamma(\alpha+1)} \int_0^t \left[h_1(s) g_1 \left(\frac{\left|(D_{0^+}^{\beta} x)(s)\right|}{s^{\alpha-\beta-1}} \right) \right. \\ & \left. + h_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{\left|\left(D_{0^+}^{\gamma} x\right)(\tau)\right|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) \right] ds,\ t > 0. \end{aligned} \tag{4.95}$$

Applying $D_{0^+}^{\beta}$ and $D_{0^+}^{\gamma}$ to both sides of (4.94), and taking Lemma 3.5 and Lemma 3.12 into account, we have

$$\begin{aligned} (D_{0^+}^{\beta} x)(t) \quad = \quad & \frac{a_1 t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{a_2 t^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} \\ & + \left(I_{0^+}^{\alpha+1-\beta} f\left(s, (D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau,\left(D_{0^+}^{\gamma} x\right)(\tau)) d\tau \right)\right)(t)\,, t > 0\,, \end{aligned}$$

$$\begin{aligned} \left(D_{0^+}^{\gamma} x\right)(t) \quad = \quad & \frac{a_1 t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} + \frac{a_2 t^{\alpha-\gamma}}{\Gamma(1+\alpha-\gamma)} \\ & + \left(I_{0^+}^{\alpha+1-\gamma} f\left(s, (D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau,\left(D_{0^+}^{\gamma} x\right)(\tau)) d\tau \right)\right)(t)\,, t > 0\,, \end{aligned}$$

respectively. Therefore for all $t > 0$,

$$\begin{aligned} & t^{1-(\alpha-\beta)} \left|(D_{0^+}^{\beta}x)(t)\right| \\ \leq\ & \frac{|a_1|}{\Gamma(\alpha-\beta)} + \frac{|a_2|\,t}{\Gamma(1+\alpha-\beta)} + \frac{t}{\Gamma(1+\alpha-\beta)} \int_0^t h_1(s) g_1\left(\frac{\left|(D_{0^+}^{\beta}x)(s)\right|}{s^{\alpha-\beta-1}}\right) ds \\ & + \frac{t}{\Gamma(1+\alpha-\beta)} \int_0^t h_2(s) g_2\left(\int_0^s h_3(\tau) g_3\left(\frac{\left|\left(D_{0^+}^{\gamma}x\right)(\tau)\right|}{\tau^{\alpha-\gamma-1}}\right) d\tau\right) ds, \quad t > 0\ . \end{aligned} \tag{4.96}$$

$$\begin{aligned} & t^{1-(\alpha-\gamma)} \left|\left(D_{0^+}^{\gamma}x\right)(t)\right| \\ \leq\ & \frac{|a_1|}{\Gamma(\alpha-\gamma)} + \frac{|a_2|\,t}{\Gamma(1+\alpha-\gamma)} + \frac{t}{\Gamma(1+\alpha-\gamma)} \int_0^t h_1(s) g_1\left(\frac{\left|(D_{0^+}^{\beta}x)(s)\right|}{s^{\alpha-\beta-1}}\right) ds \\ & + \frac{t}{\Gamma(1+\alpha-\gamma)} \int_0^t h_2(s) g_2\left(\int_0^s h_3(\tau) g_3\left(\frac{\left|\left(D_{0^+}^{\gamma}x\right)(\tau)\right|}{\tau^{\alpha-\gamma-1}}\right) d\tau\right) ds, \quad t > 0\ . \end{aligned} \tag{4.97}$$

Now, let

$$\begin{aligned} b_2 &= |a_1| \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}, \frac{1}{\Gamma(\alpha-\gamma)}\right\},\ b_3 = |a_2|\, b_4, \\ b_4 &= \max\left\{\frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(1+\alpha-\beta)}, \frac{1}{\Gamma(1+\alpha-\gamma)}\right\}, \end{aligned}$$

and

$$z(t) = b_2 + b_3 t + b_4 t \int_0^t \left[h_1(s) g_1 \left(\frac{\left|(D_{0^+}^{\beta} x)(s)\right|}{s^{\alpha-\beta-1}} \right) + h_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{\left|\left(D_{0^+}^{\gamma} x\right)(\tau)\right|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) \right] ds, \ t > 0.$$

Then, we obtain for all $t > 0$,

$$\frac{|x(t)|}{t^{\alpha-1}}, \frac{\left|(D_{0^+}^{\beta} x)(t)\right|}{t^{\alpha-\beta-1}} \text{ and } \frac{\left|\left(D_{0^+}^{\gamma} x\right)(t)\right|}{t^{\alpha-\gamma-1}} \leq z(t). \tag{4.98}$$

The remaining steps of the proof are similar to those of Theorem 4.7. ∎

CHAPTER 5

BOUNDEDNESS AND POWER-TYPE DECAY OF SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

In this chapter, we consider the fractional integro-differential equation (1.1) with the Caputo and Riemann-Liouville fractional derivatives of orders $0 \leq \beta < \mu < 1$ and $0 \leq \gamma < \mu < 1$.

The long-time behavior of solutions is investigated for the initial value problem composed of the equation (1.1) with the initial conditions $x(0) = c_0$ and $I_{0^+}^{1-\mu}x(0^+) = c_1$, where $c_0, c_1 \in \mathbb{R}$, when the fractional derivatives are of Caputo and Riemann-Liouville types, respectively. The first section is dedicated to the study of boundedness of the solutions of (1.1) for all the time with the Caputo fractional derivatives. The power-type decay of the solutions is discussed in Section 5.2 when the derivatives are the Riemann-Liouville fractional derivatives.

Throughout this chapter, we use the following classes of functions.

Definition 5.1 *We say that a function* $h : [0,\infty) \to [0,\infty)$ *is of type* $\mathcal{H}_{\sigma,\delta}$ *if* $h \in C\,[0,\infty)$ *and* $t^{\sigma}h^{\delta}(t) \in L^1\,(1,\infty)\,,\ \sigma \geq -1,\ \delta \geq 1.$

Definition 5.2 *We say that a function* $h : [0,\infty) \to [0,\infty)$ *is of type* ${}_{q}calH_{r,\eta}$ *if* $h \in C\,[0,\infty)$ *and* $t^{rq}e^{q\eta t}h^{q} \in L^1\,(0,\infty)\,,\ 0 \leq r < \frac{q-1}{q},\ \eta > 0$ *and* $q \geq 1.$

5.1 Equations with Caputo Fractional Derivatives

The equation (1.1) with the Caputo fractional derivatives of orders $0 \leq \beta, \gamma < \mu < 1$, is considered in this section, that is

$$\begin{cases} \left({}^{C}D_{0^+}^{\mu}x\right)(t) = f\left(t, ({}^{C}D_{0^+}^{\beta}x)(t), \int_0^t k\left(t, s, \left({}^{C}D_{0^+}^{\gamma}x\right)(s)\right) ds\right),\ t \geq 0, \\ x(0) = c_0, \quad c_0 \in \mathbb{R}. \end{cases} \tag{5.1}$$

We discuss the boundedness of the continuable solutions of (5.1) in the space $C^{\mu}[0,\infty)$ defined by

$$C^{\mu}[0,\infty) := \big\{f : [0,\infty) \to \mathbb{R} \;\big|\; \big| \; | \; {\scriptstyle |f,\ {}^{C}D^{\mu}_{0+} f \in C[0,\infty)\, .}$$

In addition to the hypotheses ($\mathbf{H}_1$) and ($\mathbf{H}_2$), introduced in Chapter 4, Section 4.4, we suppose that the functions f and k satisfy the following assumptions:

($\mathbf{H}_{11}$) There are functions h_1, h_2 of type $\mathcal{H}_{\mu q-1,q}$, h_3 of type $\mathcal{H}_{0,1}$; $q \geq 1$, and g_i, $i = 1,2,3$, are of type $\mathcal{G}$ with $g_1^q \propto g_2^q \propto g_3$ such that

$$|f(t,u,v)| \leq h_1(t)g_1\left(t^{\beta}|u|\right) + h_2(t)g_2\left(|v|\right), \quad (t,u,v) \in D,$$

$$|k(t,s,u)| \leq h_3(s)g_3\left(s^{\gamma}|u|\right), \ (t,s,u) \in E,$$

$$\int_{t_0}^{\infty} \frac{\tau^{q-1}d\tau}{g_1^q(\tau)} = \infty, \int_{t_0}^{\infty} \frac{d\tau}{g_2^q(\tau)} = \infty, \ \int_{t_0}^{\infty} \frac{\tau^{q-1}d\tau}{g_3(\tau)} = \infty, \ t_0 > 0.$$

The main result of this section is stated in the following theorem.

Theorem 5.1 *Suppose that the functions f and k satisfy ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_{11}$). Then, there exists a positive constant $c \in \mathbb{R}$ such that any continuable solution $x \in C^{\mu}[0,\infty)$ of problem (5.1) satisfies*

$$|x(t)| \leq c, \ \left|(D^{\beta}_{0+}x)(t)\right| \leq ct^{-\beta} \text{ and } \left|\left(D^{\gamma}_{0+}x\right)(t)\right| \leq ct^{-\gamma} \text{ for all } t > 0.$$

Proof. Applying $I_{0^+}^{\mu}$ to both sides of the equation in (5.1), we obtain

$$x(t) = c_0 + \left(I_{0^+}^{\mu} f\left(s, ({}^C D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau, \left({}^C D_{0^+}^{\gamma} x\right)(\tau)) d\tau \right)\right)(t), \qquad (5.2)$$

for all $t > 0$. Taking the derivatives ${}^C D_{0^+}^{\beta}$ and ${}^C D_{0^+}^{\gamma}$ of (5.2), gives in view of Lemma 3.13 and Lemma 3.5,

$$({}^C D_{0^+}^{\beta} x)(t) = I_{0^+}^{\mu-\beta} f\left(s, ({}^C D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau, \left({}^C D_{0^+}^{\gamma} x\right)(\tau)) d\tau \right)(t),$$

$$\left({}^C D_{0^+}^{\gamma} x\right)(t) = \left(I_{0^+}^{\mu-\gamma} f\left(s, ({}^C D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau, \left({}^C D_{0^+}^{\gamma} x\right)(\tau)) d\tau \right)\right)(t).$$

By virtue of $(\mathbf{H}_{11})$, we have

$$\begin{aligned} |x(t)| \;\leq\; & |c_0| + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} s^r \left[\tilde{h}_1(s) g_1\left(s^{\beta}\left|({}^C D_{0^+}^{\beta} x)(s)\right|\right) \right. \\ & \left. + \tilde{h}_2(s) g_2\left(\int_0^s h_3(\tau) g_3\left(\tau^{\gamma}\left|\left({}^C D_{0^+}^{\gamma} x\right)(\tau)\right|\right) d\tau \right) \right] ds, \end{aligned} \qquad (5.3)$$

for all $t > 0$, where $\tilde{h}_i(t) = t^{-r} h_i(t)$, $i = 1, 2$, $r = 1 - \mu - \frac{1}{p}$, $1 - \mu + \beta < \frac{1}{p}$, $1 - \mu + \gamma < \frac{1}{p}$ and $p = \frac{q}{q-1}$. Similarly, we get for all $t > 0$,

$$\begin{aligned} \left|({}^C D_{0^+}^{\beta} x)(t)\right| \;\leq\; & \frac{1}{\Gamma(\mu-\beta)} \int_0^t (t-s)^{\mu-\beta-1} s^r \left[\tilde{h}_1(s) g_1\left(s^{\beta}\left|({}^C D_{0^+}^{\beta} x)(s)\right|\right) \right. \\ & \left. + \tilde{h}_2(s) g_2\left(\int_0^s h_3(\tau) g_3\left(\tau^{\gamma}\left|\left({}^C D_{0^+}^{\gamma} x\right)(\tau)\right|\right) d\tau \right) \right] ds, \end{aligned}$$

$$\begin{aligned}\left|\left({}^{C}D_{0+}^{\gamma}x\right)(t)\right| \;\; &\leq \;\; \frac{1}{\Gamma(\mu-\gamma)}\int_0^t (t-s)^{\mu-\gamma-1}s^r\left[\tilde{h}_1(s)g_1\left(s^\beta\left|({}^{C}D_{0+}^{\beta}x)(s)\right|\right)\right. \\ &\left. + \tilde{h}_2(s)g_2\left(\int_0^s h_3(\tau)g_3\left(\tau^\gamma\left|\left({}^{C}D_{0+}^{\gamma}x\right)(\tau)\right|\right)d\tau\right)\right]ds.\end{aligned}$$

Using Hölder's inequality, we find

$$\begin{aligned}&\int_0^t (t-s)^{\mu-1}s^r\tilde{h}_1(s)g_1\left(s^\beta\left|({}^{C}D_{0+}^{\beta}x)(s)\right|\right)ds \\ &\leq \left(\int_0^t (t-s)^{p(\mu-1)}s^{pr}ds\right)^{\frac{1}{p}}\left(\int_0^t \tilde{h}_1^q(s)g_1^q\left(s^\beta\left|({}^{C}D_{0+}^{\beta}x)(s)\right|\right)ds\right)^{\frac{1}{q}},\ t>0.\end{aligned}$$

Notice that

$$\int_0^t (t-s)^{p(\mu-1)}s^{pr}ds = \left(I_{0+}^{p(\mu-1)+1}s^{pr}\right)(t).$$

Since $p(\mu-1)+1>0$ and $pr+1>0$, we see, from Lemma 3.5, that

$$\int_0^t (t-s)^{p(\mu-1)}s^{pr}ds = \frac{\Gamma(pr+1)}{\Gamma(p(\mu+r-1)+2)}t^{p(\mu+r-1)+1},\ \ t>0.$$

Therefore,

$$\begin{aligned}&\int_0^t (t-s)^{\mu-1}s^r\tilde{h}_1(s)g_1\left(s^\beta\left|({}^{C}D_{0+}^{\beta}x)(s)\right|\right)ds \\ &\leq B_1 t^{\mu+r-1+\frac{1}{p}}\left(\int_0^t \tilde{h}_1^q(s)g_1^q\left(s^\beta\left|({}^{C}D_{0+}^{\beta}x)(s)\right|\right)ds\right)^{\frac{1}{q}},\end{aligned}$$

where $B_1 = \left(\frac{\Gamma(pr+1)}{\Gamma(p(\mu+r-1)+2)}\right)^{\frac{1}{p}}$. Also,

$$\begin{aligned}
&\int_0^t (t-s)^{\mu-1} s^r \tilde{h}_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| \left({}^C D^\gamma_{0^+} x\right)(\tau)\right|\right) d\tau \right) ds \\
&\leq B_1 t^{\mu+r-1+\frac{1}{p}} \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| \left({}^C D^\gamma_{0^+} x\right)(\tau)\right|\right) d\tau \right) ds \right)^{\frac{1}{q}}.
\end{aligned}$$

Recalling that $\mu + r - 1 + \frac{1}{p} = 0$, so (5.3) becomes

$$\begin{aligned}
|x(t)| \;\leq\; & |c_0| + \frac{B_1}{\Gamma(\mu)} \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C D^\beta_{0^+} x)(s) \right| \right) ds \right)^{\frac{1}{q}} \\
& + \frac{B_1}{\Gamma(\mu)} \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| \left({}^C D^\gamma_{0^+} x\right)(\tau)\right|\right) d\tau \right) ds \right)^{\frac{1}{q}} \qquad (5.4)
\end{aligned}$$

for all $t > 0$. Similarly, we get the following estimates on ${}^C D^\beta_{0^+} x$ and ${}^C D^\gamma_{0^+} x$ for all $t > 0$,

$$\begin{aligned}
\left| ({}^C D^\beta_{0^+} x)(t) \right| \;\leq\; & B_2 t^{-\beta} \left[\left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C D^\beta_{0^+} x)(s) \right| \right) ds \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| \left({}^C D^\gamma_{0^+} x\right)(\tau)\right|\right) d\tau \right) ds \right)^{\frac{1}{q}} \right],
\end{aligned} \qquad (5.5)$$

$$\begin{aligned}
\left| \left({}^C D^\gamma_{0^+} x\right)(t) \right| \;\leq\; & B_3 t^{-\gamma} \left[\left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C D^\beta_{0^+} x)(s) \right| \right) ds \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| \left({}^C D^\gamma_{0^+} x\right)(\tau)\right|\right) d\tau \right) ds \right)^{\frac{1}{q}} \right],
\end{aligned} \qquad (5.6)$$

where $B_2 = \frac{1}{\Gamma(\mu-\beta)}\left(\frac{\Gamma(pr+1)}{\Gamma(p(\mu-\beta+r-1)+2)}\right)^{\frac{1}{p}}$ and $B_3 = \frac{1}{\Gamma(\mu-\gamma)}\left(\frac{\Gamma(pr+1)}{\Gamma(p(\mu-\gamma+r-1)+2)}\right)^{\frac{1}{p}}$.

Let

$$\begin{aligned} z(t) &= |c_0| + B_4\left(\int_0^t \tilde{h}_1^q(s) g_1^q\left(s^\beta \left|({}^C D_{0+}^\beta x)(s)\right|\right) ds\right)^{\frac{1}{q}} \\ &\quad + B_4\left(\int_0^t \tilde{h}_2^q(s) g_2^q\left(\int_0^s h_3(\tau) g_3\left(\tau^\gamma \left|\left({}^C D_{0+}^\gamma x\right)(\tau)\right|\right) d\tau\right) ds\right)^{\frac{1}{q}}, \end{aligned} \tag{5.7}$$

for all $t > 0$, where

$$B_4 = \max\left\{\frac{B_1}{\Gamma(\mu)}, B_2, B_3\right\}.$$

It is obvious from the inequalities (5.4), (5.5), (5.6) and (5.7) that

$$|x(t)| \le z(t),\ t^\beta \left|({}^C D_{0+}^\beta x)(t)\right| \le z(t),\ t^\gamma \left|\left({}^C D_{0+}^\gamma x\right)(t)\right| \le z(t),\ t > 0. \tag{5.8}$$

As g_1, g_2 and g_3 are nondecreasing functions, it clear that

$$\begin{aligned} z(t) &= |c_0| + B_4\left(\int_0^t \tilde{h}_1^q(s) g_1^q(z(s))\, ds\right)^{\frac{1}{q}} \\ &\quad + B_4\left(\int_0^t \tilde{h}_2^q(s) g_2^q\left(\int_0^s h_3(\tau) g_3(z(\tau))\, d\tau\right) ds\right)^{\frac{1}{q}},\ t > 0. \end{aligned}$$

Using Lemma 3.19, we obtain

$$\begin{aligned} (z(t))^q &\le B_5 + B_6 \int_0^t \tilde{h}_1^q(s) g_1^q(z(s))\, ds \\ &\quad + B_6 \int_0^t \tilde{h}_2^q(s) g_2^q\left(\int_0^s h_3(\tau) g_3(z(\tau))\, d\tau\right) ds, \quad t > 0, \end{aligned}$$

where $B_5 = 3^{q-1}|c_0|^q$ and $B_6 = 3^{q-1} B_4^q$. Let $u(t) = (z(t))^q$, then we have for all

$t > 0$,

$$u(t) \leq B_5 + B_6 \int_0^t \tilde{h}_1^q(s) g_1^q(u^{\frac{1}{q}}(s))ds + B_6 \int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3(u^{\frac{1}{q}}(\tau))d\tau \right) ds.$$

From Lemma 3.24, with

$$\lambda_1(t) = B_6\tilde{h}_1^q(t),\ \lambda_2(t) = B_6\tilde{h}_2^q(t),\ \lambda_3(t) = h_3(t),$$

$$w_1(u) = g_1^q\left(u^{\frac{1}{q}}\right),\ w_3(u) = g_3\left(u^{\frac{1}{q}}\right),$$
$$w_2\left(\int_0^s h_3(\tau)w_3(u(\tau))d\tau\right) = g_2^q\left(\int_0^s h_3(\tau)g_3(u^{\frac{1}{q}}(\tau))d\tau\right),$$

we deduce that

$$\begin{aligned} u(t) &\leq W_3^{-1}\left(W_3(c_2) + \int_0^t \lambda_3(s)ds\right) \\ &\leq W_3^{-1}\left(W_3(c_2) + \int_0^\infty \lambda_3(s)ds\right) := M \quad \text{for all } t > 0, \end{aligned}$$

where M is a positive constant. The desired result follows in virtue of (5.8). ∎

Remark 5.1 *As special case of Theorem 5.1 with $\beta = \gamma = 0$, there exists a positive constant $c \in \mathbb{R}$ such that any continuable solution $x \in C^\mu[0,\infty)$ of the problem*

$$\begin{cases} \left({}^CD_{0^+}^\mu x\right)(t) = f\left(t, x(t), \int_0^t k(t,s,x(s))\,ds\right),\ t \geq 0, \\ x(0) = c_0, \quad c_0 \in \mathbb{R}, \end{cases}$$

satisfies $|x(t)| \leq c$, for all $t > 0$.

Remark 5.2 *It is not hard to see that the conclusion of Theorem 5.1 is still valid* *if we replace the condition ($\boldsymbol{H}_{11}$) by the following condition: There are functions k_1 of type $\mathcal{H}_{\mu q-1,q}$, k_2 of type $\mathcal{H}_{0,1}$, $q \geq 1$ and f_i of type $\mathcal{G}$ with $f_1^q f_2^q \propto f_3$ such that*

$$|f(t,u,v)| \leq k_1(t) f_1(t^\beta |u|) f_2(|v|), \quad (t,u,v) \in D,$$

$$|k(t,s,u)| \leq k_2(s) f_3(s^\gamma |u|), \ (t,s,u) \in E,$$

$$\int_{t_0}^{\infty} \frac{\tau^{q-1} d\tau}{f_1^q(\tau) f_2^q(\tau^q)} = \infty, \ \int_{t_0}^{\infty} \frac{\tau^{q-1} d\tau}{f_3(\tau)} = \infty, \ t_0 > 0.$$

5.2 Equations with Riemann-Liouville Fractional Derivatives

Here we consider the following problem

$$\begin{cases} \left(D_{0^+}^{\mu} x\right)(t) = f\left(t, (D_{0^+}^{\beta} x)(t), \int_0^t k(t,s,\left(D_{0^+}^{\gamma} x\right)(s)) ds\right), \quad t > 0, \\ \left(I_{0^+}^{1-\mu} x\right)(0^+) = c_1, \ c_1 \in \mathbb{R}, \end{cases} \tag{5.9}$$

where $D_{0^+}^{\mu}$, $D_{0^+}^{\beta}$, $D_{0^+}^{\gamma}$ are the Riemann-Liouville fractional derivative of orders μ, β and γ, respectively, with $0 \leq \beta < \mu < 1$ and $0 \leq \gamma < \mu < 1$.

We study the power-type decay of continuable solutions for the problem (5.9) in the space $C_{1-\mu}^{\mu}[0,\infty)$ defined in (4.65). Before stating and proving our next theorem, we assume that the functions f and k satisfy the following:

($\mathbf{H}_{12}$) There are functions h_1, h_2 of type ${}_q\mathcal{H}_{r,\eta}$, h_3 of type $\mathcal{H}_0$ and g_i of type $\mathcal{G}$,

$i = 1, 2, 3$ with $g_1^q \propto g_2^q \propto g_3$ $\;0 \le r < \frac{q-1}{q}$, $\eta > 0$ and $q \ge 1$ such that

$$\begin{aligned} |f(t,u,v)| &\le h_1(t) g_1\left(\frac{|u|}{t^{\mu-\beta-1}}\right) + h_2(t) g_2\left(|v|\right), \ (t,u,v) \in D, \\ |k(t,s,u)| &\le h_3(s) g_3\left(\frac{|u|}{s^{\mu-\gamma-1}}\right), \quad (t,s,u) \in E, \quad 0 \le \beta, \gamma < \mu < 1, \end{aligned}$$

$$\int_{t_0}^{\infty} \frac{\tau^{q-1} d\tau}{g_1^q(\tau)} = \infty, \quad \int_{t_0}^{\infty} \frac{d\tau}{g_2^q(\tau)} = \infty, \quad \int_{t_0}^{\infty} \frac{\tau^{q-1} d\tau}{g_3(\tau)} = \infty, \ t_0 > 0.$$

Theorem 5.2 *Suppose that the functions f and k satisfy the conditions ($\tilde{\boldsymbol{H}}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_{12}$). Then, there exists a positive constant c such that any continuable solution $x \in C_{1-\mu}^{\mu}[0,\infty)$ of the problem (5.9) satisfies*

$$|x(t)| \le ct^{\mu-1}, \quad \left|(D_{0^+}^{\beta} x)(t)\right| \le ct^{\mu-\beta-1} \text{ and } \left|\left(D_{0^+}^{\gamma} x\right)(t)\right| \le ct^{\mu-\gamma-1} \text{ for all } \ t > 0.$$

Proof. Applying $I_{0^+}^{\mu}$ to both sides of the equation in (5.9) gives

$$x(t) = \frac{c_1 t^{\mu-1}}{\Gamma(\mu)} + \left(I_{0^+}^{\mu} f\left(s, (D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau,\left(D_{0^+}^{\gamma} x\right)(\tau)) d\tau\right)\right)(t), \ t \ge 0. \tag{5.10}$$

Lemma 3.5 and Lemma 3.12 allow us to write,

$$(D_{0^+}^{\beta} x)(t) = \frac{c_1 t^{\mu-\beta-1}}{\Gamma(\mu-\beta)} + \left(I_{0^+}^{\mu-\beta} f\left(s, (D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau,\left(D_{0^+}^{\gamma} x\right)(\tau)) d\tau\right)\right)(t), \tag{5.11}$$

$$\left(D_{0^+}^{\gamma} x\right)(t) = \frac{c_1 t^{\mu-\gamma-1}}{\Gamma(\mu-\gamma)} + \left(I_{0^+}^{\mu-\gamma} f\left(s, (D_{0^+}^{\beta} x)(s), \int_0^s k(s,\tau,\left(D_{0^+}^{\gamma} x\right)(\tau)) d\tau\right)\right)(t), \tag{5.12}$$

for all $t > 0$. In virtue of the condition $(\mathbf{H}_{12})$, we observe that for all $t > 0$,

$$\begin{aligned}
|x(t)| \leq\ & \frac{|c_1|\, t^{\mu-1}}{\Gamma(\mu)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} s^{-r} e^{-\eta s} \left[\tilde{h}_1(s) g_1 \left(\frac{\left| (D_{0^+}^{\beta} x)(s) \right|}{s^{\mu-\beta-1}} \right) \right. \\
& \left. + \tilde{h}_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{\left| \left(D_{0^+}^{\gamma} x \right)(\tau) \right|}{\tau^{\mu-\gamma-1}} \right) d\tau \right) ds \right]. \qquad (5.13)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\left| (D_{0^+}^{\beta} x)(t) \right| \leq\ & \frac{|c_1|\, t^{\mu-\beta-1}}{\Gamma(\mu-\beta)} + \frac{1}{\Gamma(\mu-\beta)} \int_0^t (t-s)^{\mu-\beta-1} s^{-r} e^{-\eta s} \tilde{h}_1(s) \\
& \times g_1 \left(\frac{\left| (D_{0^+}^{\beta} x)(s) \right|}{s^{\mu-\beta-1}} \right) ds + \frac{1}{\Gamma(\mu-\beta)} \int_0^t (t-s)^{\mu-\beta-1} s^{-r} e^{-\eta s} \tilde{h}_2(s) \\
& \times g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{\left| \left(D_{0^+}^{\gamma} x \right)(\tau) \right|}{\tau^{\mu-\gamma-1}} \right) d\tau \right) ds, \qquad (5.14)
\end{aligned}$$

and

$$\begin{aligned}
\left| \left(D_{0^+}^{\gamma} x \right)(t) \right| \leq\ & \frac{|c_1|\, t^{\mu-\gamma-1}}{\Gamma(\mu-\gamma)} + \frac{1}{\Gamma(\mu-\gamma)} \int_0^t (t-s)^{\mu-\gamma-1} s^{-r} e^{-\eta s} \tilde{h}_1(s) \\
& \times g_1 \left(\frac{\left| (D_{0^+}^{\beta} x)(s) \right|}{s^{\mu-\beta-1}} \right) ds + \frac{1}{\Gamma(\mu-\gamma)} \int_0^t (t-s)^{\mu-\gamma-1} s^{-r} e^{-\eta s} \tilde{h}_2(s) \\
& \times g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{\left| \left(D_{0^+}^{\gamma} x \right)(\tau) \right|}{\tau^{\mu-\gamma-1}} \right) d\tau \right) ds, \qquad (5.15)
\end{aligned}$$

where $\tilde{h}_i(t) = t^r e^{\eta t} h_i(t)$, $i = 1, 2$, $\eta > 0, 1 - pr > 0$, $\frac{1}{p} > \max\{1-\mu+\beta, 1-\mu+\gamma\}$ and $p = \frac{q}{q-1}$.

Using Hölder's inequality and Lemma 3.18, the estimates (5.13) – (5.15) become

$$\begin{aligned}\frac{|x(t)|}{t^{\mu-1}} \quad \leq \quad & \frac{|c_1|}{\Gamma(\mu)}+\frac{C^{\frac{1}{p}}}{\Gamma(\mu)}\left(\int_0^t \tilde{h}_1^q(s) g_1^q\left(\frac{\left|(D_{0^+}^{\beta} x)(s)\right|}{s^{\mu-\beta-1}}\right) ds\right)^{\frac{1}{q}} \\ & +\frac{C^{\frac{1}{p}}}{\Gamma(\mu)}\left(\int_0^t \tilde{h}_2^q(s) g_2^q\left(\int_0^s h_3(\tau) g_3\left(\frac{\left|\left(D_{0^+}^{\gamma} x\right)(\tau)\right|}{\tau^{\mu-\gamma-1}}\right) d\tau\right) ds\right)^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\frac{\left|(D_{0^+}^{\beta} x)(t)\right|}{t^{\mu-\beta-1}} \quad \leq \quad & \frac{|c_1|}{\Gamma(\mu-\beta)}+\frac{C_1^{\frac{1}{p}}}{\Gamma(\mu-\beta)}\left(\int_0^t \tilde{h}_1^q(s) g_1^q\left(\frac{\left|(D_{0^+}^{\beta} x)(s)\right|}{s^{\mu-\beta-1}}\right) ds\right)^{\frac{1}{q}} \\ & +\frac{C_1^{\frac{1}{p}}}{\Gamma(\mu-\beta)}\left(\int_0^t \tilde{h}_2^q(s) g_2^q\left(\int_0^s h_3(\tau) g_3\left(\frac{\left|\left(D_{0^+}^{\gamma} x\right)(\tau)\right|}{\tau^{\mu-\gamma-1}}\right) d\tau\right) ds\right)^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\frac{\left|\left(D_{0^+}^{\gamma} x\right)(t)\right|}{t^{\mu-\gamma-1}} \quad \leq \quad & \frac{|c_1|}{\Gamma(\mu-\gamma)}+\frac{C_2^{\frac{1}{p}}}{\Gamma(\mu-\gamma)}\left(\int_0^t \tilde{h}_1^q(s) g_1^q\left(\frac{\left|(D_{0^+}^{\beta} x)(s)\right|}{s^{\mu-\beta-1}}\right) ds\right)^{\frac{1}{q}} \\ & +\frac{C_2^{\frac{1}{p}}}{\Gamma(\mu-\gamma)}\left(\int_0^t \tilde{h}_2^q(s) g_2^q\left(\int_0^s h_3(\tau) g_3\left(\frac{\left|\left(D_{0^+}^{\gamma} x\right)(\tau)\right|}{\tau^{\mu-\gamma-1}}\right) d\tau\right) ds\right),\end{aligned}$$

where

$$C=\max\left\{1, 2^{p(1-\mu)}\right\} \Gamma(1-pr)\left(1+\frac{(1-pr)(2-pr)}{p(\mu-1)+1}\right)(p\eta)^{pr-1}.$$

$$C_1=\max\left\{1, 2^{p(1-\mu-\beta)}\right\} \Gamma(1-pr)\left(1+\frac{(1-pr)(2-pr)}{p(\mu-\beta-1)+1}\right)(p\eta)^{pr-1},$$

$$C_2=\max\left\{1, 2^{p(1-\mu-\gamma)}\right\} \Gamma(1-pr)\left(1+\frac{(1-pr)(2-pr)}{p(\mu-\gamma-1)+1}\right)(p\eta)^{pr-1}.$$

Defining

$$
\begin{aligned}
z(t) \;&=\; A_1 + A_2 \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(\frac{\left| (D_{0^+}^{\beta} x)(s) \right|}{s^{\mu-\beta-1}} \right) ds \right)^{\frac{1}{q}} \\
&\quad + A_2 \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\frac{\left| \left(D_{0^+}^{\gamma} x \right)(\tau) \right|}{\tau^{\mu-\gamma-1}} \right) d\tau \right) ds \right)^{\frac{1}{q}}, \quad t > 0,
\end{aligned}
$$

and using

$$
\frac{|x(t)|}{t^{\mu-1}}, \; \frac{\left| (D_{0^+}^{\beta} x)(t) \right|}{t^{\mu-\beta-1}}, \; \frac{\left| \left(D_{0^+}^{\gamma} x \right)(t) \right|}{t^{\mu-\gamma-1}} \leq z(t), \text{ for all } t > 0, \tag{5.16}
$$

we arrive at

$$
\begin{aligned}
z(t) \;&=\; A_1 + A_2 \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(z(s) \right) ds \right)^{\frac{1}{q}} \\
&\quad + A_2 \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(z(\tau) \right) d\tau \right) ds \right)^{\frac{1}{q}}, \quad t > 0,
\end{aligned}
$$

where

$$
A_1 = |c_1| \max \left\{ \frac{1}{\Gamma(\mu)}, \frac{1}{\Gamma(\mu-\beta)}, \frac{1}{\Gamma(\mu-\gamma)} \right\},
$$

$$
A_2 = |c_1| \max \left\{ \frac{C^{\frac{1}{p}}}{\Gamma(\mu)}, \frac{C_1^{\frac{1}{p}}}{\Gamma(\mu-\beta)}, \frac{C_2^{\frac{1}{p}}}{\Gamma(\mu-\gamma)} \right\}.
$$

Taking the power $q \geq 1$ of both side and using Lemma 3.19 with $m = 3$, we obtain

$$
u(t) \leq A_3 + A_4 \int_0^t \tilde{h}_1^q(s) g_1^q(u^{\frac{1}{q}}(s)) ds + A_4 \int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3(u^{\frac{1}{q}}(\tau)) d\tau \right) ds,
$$

for all $t > 0$, where $u(t) = z^q(t)$, $A_3 = 3^{q-1} A_1^q$, $A_4 = 3^{q-1} A_2^q$.

Now, we conclude from Lemma 3.24, that there exists a positive constant M_1 such that

$$u(t) \leq M_1 \text{ for all } t > 0.$$

Hence, $z(t) \leq c := M_1^{\frac{1}{q}}$ and as a result of inequality (5.16), the assertion of the theorem is established. ∎

Example 5.1 *Consider the equation*

$$\left(D_{0^+}^{\mu} x\right)(t) = e^{-t} \ (x(t))^{\frac{1}{3}} + e^{-t} \left(\int_0^t \frac{s^{\lambda} e^{-t}}{(t^2+1) e^s} x(s) ds \right)^{\frac{1}{3}}, \quad t > 0, \tag{5.17}$$

where $0 < \mu < 1$ and $\lambda > -\mu - 1$. The right-hand side of the equation (5.17) can be rewritten as

$$t^{\frac{1}{3}(\mu-1)} e^{-t} \ \frac{(x(t))^{\frac{1}{3}}}{t^{\frac{1}{3}(\mu-1)}} + e^{-t} \left(\int_0^t s^{\lambda+\mu-1} e^{-(s+t)} \frac{x(s)}{s^{\mu-1}} ds \right)^{\frac{1}{3}}, \ t > 0.$$

Let

$$h_1(t) = t^{\frac{1}{3}(\mu-1)} e^{-\eta_1 t}, \ h_2(t) = e^{-\eta_2 t}, \ h_3(t) = t^{\lambda+\mu-1} e^{-\eta_3 t}, \ \lambda > -\mu, \ t > 0,$$

$$0 < \eta < \eta_i \leq 1, \ i = 1,2,3, \ g_1(t) = \ g_2(t) = t^{\frac{1}{3}} \text{ and } g_3(t) = t, \ t > 0.$$

Clearly, these functions satisfy the condition ($\boldsymbol{H}_{12}$) with $\beta = \gamma = 0$.

CHAPTER 6

NONEXISTENCE OF GLOBAL SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

The aim of this chapter is to study the nonexistence of (nontrivial) global solutions for the initial value problem formed by (1.1) subject to some appropriate initial conditions when

$$\begin{aligned} & f\left(t, (D^{\beta}_{0^+}x)(t), \int_0^t k(t,s,\left(D^{\gamma}_{0^+}x\right)(s))ds\right) \\ & \geq -\sigma (D^{\beta}_{0^+}x)(t) + \int_0^t h(t-s)\,|x(s)|^q\,ds, \end{aligned}$$

for some nonnegative function h. The constant σ could be any nonnegative real number. For the purpose of studying the effect of considering one or two fractional derivatives, we choose σ to be either 0 or 1.

Our objective is to establish some criteria under which there are no (nontrivial) global solutions in some specific underlying space. To the best of our knowledge there are no investigations on the nonexistence of solutions for such fractional integro-differential inequalities. This chapter is organized as follows: In the next section we introduce a test function and derive some of its properties. Section 6.2 is devoted to the statements and proofs of our results when the fractional derivatives are of Caputo type. Examples for special types of kernels are provided. In Section 6.3 we study the nonexistence of global nontrivial solutions for problems with the Riemann-Liouville fractional derivatives.

6.1 The Test Function, Some Identities and Inequalities

In this section, we consider the test function

$$\varphi(t) := \begin{cases} T^{-\lambda}(T-t)^{\lambda}, & 0 \le t \le T, \\ 0, & t > T. \end{cases} \tag{6.1}$$

This function enjoys the following properties.

Lemma 6.1 *Let φ be as in* (6.1) *and $r > 1$, then, for $\lambda > nr - 1$, $n = 0, 1, 2, \ldots$,* *we have*

$$\int_0^T \varphi^{1-r}(t)\,|D^n\varphi(t)|^r\,dt = C_{n,r}T^{1-nr},\ T>0,$$

where

$$C_{n,r} = \frac{(\Gamma(\lambda+1))^r}{(\lambda-nr+1)\,(\Gamma(\lambda-n+1))^r}\ .$$

Proof. Since the nth derivative of φ is

$$\begin{aligned} D^n\varphi(t) &= (-1)^n\,\lambda\,(\lambda-1)\,(\lambda-2)\ldots(\lambda-n+1)\,T^{-\lambda}\,(T-t)^{\lambda-n} \\ &= \frac{(-1)^n\,\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}T^{-\lambda}\,(T-t)^{\lambda-n},\ \text{for all}\ \ 0\le t\le T, \end{aligned}$$

we deduce that

$$\begin{aligned} \int_0^T \varphi^{1-r}(t)\,|D^n\varphi(t)|^r\,dt &= \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}\right)^r T^{-\lambda}\int_0^T (T-t)^{\lambda-nr}\,dt \\ &= C_{n,r}T^{1-nr}, \quad \text{for all}\ \ 0\le t\le T. \end{aligned}$$

∎

Lemma 6.2 *Let $\alpha\ge 0$ and φ be as in* (6.1) *with $\lambda>\alpha-1$, then, we have for all $0\le t\le T$,*

$$\left(D^\alpha_{T^-}\varphi\right)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}T^{-\lambda}\,(T-t)^{\lambda-\alpha}\,, \tag{6.2}$$

$$\int_0^T t^m\left(D^\alpha_{T^-}\varphi\right)(t)\,dt = G_{m,\lambda}T^{m+1-\alpha},\quad m=0,1,2,...,n-1,\ n=[\alpha]+1, \tag{6.3}$$

where $G_{m,\lambda} = \frac{(-1)^m m!\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+m+2)}$.

Proof. For all $0 \leq t \leq T$, we have from Lemma 3.5,

$$\left(D^{\alpha}_{T^-}\varphi\right)(t) = \left(D^{\alpha}_{T^-}T^{-\lambda}\left(T-s\right)^{\lambda}\right)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}T^{-\lambda}\left(T-t\right)^{\lambda-\alpha}.$$

For $m = 0$, it is obvious that

$$\int_0^T \left(D^{\alpha}_{T^-}\varphi\right)(t)\,dt = \frac{\Gamma(\lambda+1)}{\Gamma\left(\lambda-\alpha+2\right)}T^{1-\alpha}.$$

For $m = 1, 2, ..., n-1$, an integration by parts m times over $[0, T]$, gives

$$\begin{aligned}\int_0^T t^m \left(D^{\alpha}_{T^-}\varphi\right)(t)\,dt &= \sum_{j=0}^{m-1}\left[(-1)^j \frac{m!}{(m-j)!}t^{m-j}\left(I^{j+1}_{T^-}D^{\alpha}_{T^-}\varphi\right)(t)\right]_0^T \\ &\quad + (-1)^m m! \int_0^T \left(I^m_{T^-}D^{\alpha}_{T^-}\varphi\right)(t)\,dt. \qquad (6.4)\end{aligned}$$

Using (6.2) and Lemma 3.5, we find that for all $0 \leq t \leq T$,

$$\left(I^{j+1}_{T^-}D^{\alpha}_{T^-}\varphi\right)(t) = \frac{\Gamma(\lambda+1)}{\Gamma\left(\lambda-\alpha+j+2\right)}T^{-\lambda}\left(T-t\right)^{\lambda-\alpha+j+1},$$

$$\left(I^m_{T^-}D^{\alpha}_{T^-}\varphi\right)(t) = \frac{\Gamma(\lambda+1)}{\Gamma\left(\lambda-\alpha+m+1\right)}T^{-\lambda}\left(T-t\right)^{\lambda-\alpha+m}.$$

Therefore

$$\left[t^{m-j}\left(I^{j+1}_{T^-}D^{\alpha}_{T^-}\varphi\right)(t)\right]_0^T = 0 \text{ for all } j = 0, 1, 2, ..., m-1, \qquad (6.5)$$

and

$$\int_0^T \left(I_{T^-}^m D_{T^-}^{\alpha}\varphi\right)(t)\,dt = \frac{\Gamma(\lambda+1)}{\Gamma\left(\lambda-\alpha+m+2\right)} T^{m-\alpha+1}. \tag{6.6}$$

Now, substituting (6.5) and (6.6) in (6.4) gives the desired result in (6.3). ▮

Lemma 6.3 *Let $\alpha \geq 0$, $n = [\alpha]+1$ and φ be as in (6.1) with $\lambda > \alpha - 1$, then*

$$\left(I_{T^-}^{n-\alpha}\varphi\right)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+n-\alpha+1)} T^{-\lambda}\left(T-t\right)^{\lambda+n-\alpha},$$

for all $0 \leq t \leq T$. Moreover, $I_{T^-}^{n-\alpha}\varphi \in AC^n[0,T]$.

Proof. It follows from Lemma 3.5 that

$$\left(I_{T^-}^{n-\alpha}\varphi\right)(t) = \left(I_{T^-}^{n-\alpha}\left(T^{-\lambda}\left(T-s\right)^{\lambda}\right)\right)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+n-\alpha+1)} T^{-\lambda}\left(T-t\right)^{\lambda+n-\alpha},$$

for all $0 \leq t \leq T$, which is clearly in the space $AC^n[0,T]$ for $\lambda > \alpha-1$. ▮

Lemma 6.4 *Let $\alpha \geq 0$ and $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ for $\alpha \in \mathbb{N}_0$. For $g \in C[a,b]$ and $f, I_{b^-}^{n-\alpha}g \in AC^n[a,b]$, we have*

$$\int_a^b g(t)\left({}^C D_{a^+}^{\alpha} f\right)(t)\,dt = \int_a^b f(t)\left(D_{b^-}^{\alpha} g\right)(t)\,dt + \sum_{j=0}^{n-1}\left[\left(D_{b^-}^{\alpha+j-n} g\right)(t)\left(D^{n-1-j} f\right)(t)\right]_a^b.$$

Proof. As $f \in AC^n[a,b]$, we obtain from the definition (3.6),

$$\int_a^b g(t)\left({}^C D_{a^+}^{\alpha} f\right)(t)\,dt = \int_a^b g(t)\left(I_{a^+}^{n-\alpha} D^n f\right)(t)dt.$$

Now, as $g \in L^{m_1}(a,b)$ for any $m_1 \geq 1$ and $D^n f \in L^1(a,b)$, we deduce from Lemma 3.9,

$$\int_a^b g(t)\left(I_{a^+}^{n-\alpha} D^n f\right)(t)dt = \int_a^b D^n f(t)\left(I_{b^-}^{n-\alpha} g\right)(t)dt.$$

Since $I_{b-}^{n-\alpha} g \in AC^n[a,b]$ and $D^{n-1} f \in AC[a,b]$, the integrating by parts n times yields

$$\begin{aligned}\int_a^b g(t)\left({}^C D_{a^+}^{\alpha} f\right)(t)\,dt &= \sum_{j=0}^{n-1}\left[\left(D_{b^-}^{\alpha+j-n} g\right)(t)\left(D^{n-1-j} f\right)(t)\right]_a^b \\ &\quad + (-1)^n \int_a^b f(t) D^n\left(I_{b^-}^{n-\alpha} g\right)(t)\,dt,\end{aligned}$$

which is the desired result with the help of (3.5). ∎

Lemma 6.5 *Let $\alpha \geq 0$ and φ be as in* (6.1) *with $\lambda > \max\{0, \alpha - 1\}$. Suppose that $f \in AC^n[0,T]$, $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ for $\alpha \in \mathbb{N}_0$. Then,*

$$\int_0^T \varphi(t)\left({}^C D_{0^+}^{\alpha} f\right)(t)\,dt = \int_0^T f(t)\left(D_{T^-}^{\alpha}\varphi\right)(t)\,dt - \sum_{j=0}^{n-1} \hat{G}_{j,\alpha} T^{n-\alpha-j}\left(D^{n-1-j} f\right)(0),$$

where $\hat{G}_{j,\alpha} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha-j+n)}$.

Proof. From (6.2) in Lemma 6.2, we have for $j = 0, 1, 2, ..., n-1$,

$$\left(D_{T^-}^{\alpha+j-n}\varphi\right)(0) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha-j+n)} T^{n-\alpha-j},$$

$$\left(D_{T^-}^{\alpha+j-n}\varphi\right)(T) = 0.$$

As $\varphi \in C[0,T]$ for $\lambda > 0$ and $I_{T-}^{n-\alpha}\varphi \in AC^n[0,T]$ by Lemma 6.3, the result follows from Lemma 6.4. ∎

Lemma 6.6 *Let $\rho \geq 0$, $n = [\rho]+1$ and $r > 1$. Let φ be as in (6.1) with $\lambda > nr-1$. Suppose that h is a nonnegative function which is different from zero almost everywhere and $t^{r(n-\rho-1)}h^{1-r}(t) \in L^1_{loc}[0,\infty)$. Then, for any $T > 0$,*

$$\int_0^T \left(D_{T-}^{\rho}\varphi\right)^r(t)\left(\int_t^T h(s-t)\varphi(s)\,ds\right)^{1-r} dt \leq \hat{C}_{\rho,r}T^{1-nr}\int_0^T t^{r(n-\rho-1)}h^{1-r}(t)dt,$$

where $\hat{C}_{\rho,r} = \left(\frac{1}{\Gamma(n-\rho)}\right)^r C_{n,r}$, $C_{n,r}$ is given in Lemma 6.1.

Proof. It is clear that $\varphi^{(j)}(T) = 0$ for all $j = 0,1,...,n-1$. Hence $D_{T-}^{\rho}\varphi = {}^{C}D_{T-}^{\rho}\varphi$. Also, as $\varphi \in AC^n[0,T]$ for $\lambda > n-1$, it follows that ${}^{C}D_{T-}^{\rho}\varphi = (-1)^n I_{T-}^{n-\rho}D^n\varphi$ and

$$\begin{aligned}
&\left(D_{T-}^{\rho}\varphi\right)(t) \leq \left(I_{T-}^{n-\rho}\left|D^n\varphi\right|\right)(t) = \frac{1}{\Gamma(n-\rho)}\int_t^T (s-t)^{n-\rho-1}\left|\left(D^n\varphi\right)(s)\right| ds \\
&= \frac{1}{\Gamma(n-\rho)}\int_t^T (s-t)^{n-\rho-1}h^{\frac{1}{r'}}(s-t)\varphi^{\frac{1}{r'}}(s)\,h^{-\frac{1}{r'}}(s-t)\varphi^{-\frac{1}{r'}}(s)\left|\left(D^n\varphi\right)(s)\right| ds.
\end{aligned}$$

Using Hölder's inequality with $\frac{1}{r}+\frac{1}{r'} = 1$, we find

$$\begin{aligned}
\left(D_{T-}^{\rho}\varphi\right)(t) \;&\leq\; \frac{1}{\Gamma(n-\rho)}\left(\int_t^T h(s-t)\varphi(s)\,ds\right)^{\frac{1}{r'}} \\
&\quad\times\left(\int_t^T (s-t)^{(n-\rho-1)r}\,h^{-\frac{r}{r'}}(s-t)\varphi^{-\frac{r}{r'}}(s)\left|\left(D^n\varphi\right)(s)\right|^r ds\right)^{\frac{1}{r}}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_0^T \left(D_{T^-}^{\rho}\varphi\right)^r(t)\left(\int_t^T h(s-t)\varphi(s)\,ds\right)^{1-r} dt \\
&\le b_1 \int_0^T \int_t^T (s-t)^{r(n-\rho-1)}\, h^{-\frac{r}{r'}}(s-t)\varphi^{-\frac{r}{r'}}(s)\left|\left(D^n\varphi\right)(s)\right|^r dsdt \\
&= b_1 \int_0^T \int_0^s (s-t)^{r(n-\rho-1)}\, h^{1-r}(s-t)\varphi^{1-r}(s)\left|\left(D^n\varphi\right)(s)\right|^r dtds \\
&= b_1 \int_0^T \varphi^{1-r}(s)\left|\left(D^n\varphi\right)(s)\right|^r \left(\int_0^s (s-t)^{r(n-\rho-1)}\, h^{1-r}(s-t)dt\right) ds.
\end{aligned}
$$

where $b_1 = \left(\frac{1}{\Gamma(n-\rho)}\right)^r$. Let $\tau = s-t$ in the inner integral, we obtain the uniform bound

$$
\int_0^s \tau^{r(n-\rho-1)} h^{1-r}(\tau)d\tau \le \int_0^T \tau^{r(n-\rho-1)} h^{1-r}(\tau)d\tau.
$$

Now the result follows from Lemma 6.1. ∎

6.2 Problems with Caputo Fractional Derivatives

We consider in this section the initial value problem

$$
\begin{cases}
\left({}^{C}D_{0+}^{\mu}x\right)(t) + \sigma\left({}^{C}D_{0+}^{\beta}x\right)(t) \ge \int_0^t h(t-s)\left|x(s)\right|^q ds,\ t>0,\ q>1, \\
x(0)=x_0,\ x'(0)=x_1,\ \ldots,\ x^{(n-1)}(0)=x_{n-1},
\end{cases}
\tag{6.7}
$$

where ${}^{C}D_{0+}^{\mu}$ and ${}^{C}D_{0+}^{\beta}$ are the Caputo fractional derivatives of orders μ and β, respectively, $n-1 \le \beta < \mu < n$, $n \in \mathbb{N}$, $\sigma = 0, 1$ and $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}$, are given initial data.

Clearly, $x \equiv 0$ is the trivial solution of (6.7) (in case of zero initial data). We shall exclude this situation and consider only nontrivial solutions.

By a global nontrivial solution to the initial value problem (6.7), we mean a nonzero function $x \in AC^n[0,T]$ for all $T > 0$, for which the inequality in (6.7) holds for all $t > 0$ and satisfying the initial data in (6.7).

Note that the two spaces $C^{\mu,1}[0,T]$ and $C^{\mu}[0,T]$ defined in Section 4.4 and Section 5.1, respectively, are subspaces of $AC[0,T]$ for all $T > 0$.

To the best of our knowledge there are no investigations on the nonexistence of global solutions for fractional integro-differential inequalities of type (6.7).

Observe that when $0 < \mu < 1$, $\beta = \sigma = 1$ and $h(t) = \delta(t)$ (the Dirac delta function) in (6.7) then we recover (2.18). Also, when $\mu = 1$, $\sigma = 1$, $\beta = 0$ and $h(t) = \delta(t)$, the equation corresponding to (6.7) is equivalent to (2.6). The nonlinear Volterra integro-differential equation (2.10) is a special case of (6.7) with a constant kernel and $\mu = \beta = 1$. If $\mu = \sigma = 1$ and $\beta = 0$ in (6.7) and $g(x(s)) = |x(s)|^q$ in (2.13), we have an equivalence between (6.7) and (2.13).

We prove nonexistence of (nontrivial) global solutions for the initial value problem (6.7) in the space $AC^n[0,T]$, for all $T > 0$ and under some integral conditions on the kernel h. The proof is based on the test function method due to Mitidieri and Pohozaev [184] adopted here to the fractional case.

To investigate the competition between the roles of the first order derivative and the fractional derivatives of orders $0 < \mu < 1$ and $1 \leq \mu < 2$ in determining the nonexistence criteria, we consider the following two cases.

6.2.1 Case $0 \le \mu < 1,\ \beta = 1$

In this subsection we prove nonexistence results for the problem

$$\begin{cases} \sigma x'(t) + \left({}^{C}D_{0+}^{\mu}x\right)(t) \ge \int_0^t h(t-s)\,|x(s)|^q\,ds,\ t>0,\ q>1,\ 0\le\mu<1, \\ x(0)=x_0,\ x_0\in\mathbb{R}. \end{cases} \tag{6.8}$$

We start first with the case $\sigma = 0$, that is,

$$\begin{cases} \left({}^{C}D_{0+}^{\mu}x\right)(t) \ge \int_0^t h(t-s)\,|x(s)|^q\,ds,\ t>0,\ q>1,\ 0<\mu<1, \\ x(0)=x_0,\ x_0\in\mathbb{R}. \end{cases} \tag{6.9}$$

Theorem 6.1 *Let $0<\mu<1$ and h be a nonnegative function which is different from zero almost everywhere with $t^{-\mu q'}h^{1-q'}(t) \in L^1_{loc}[0,\infty)$. Assume that,*

$$\lim_{T\to\infty} T^{1-q'} \int_0^T t^{-\mu q'} h^{1-q'}(t)dt = 0, \tag{6.10}$$

where $q' = \frac{q}{q-1}$. Then, the problem (6.9) does not admit any global nontrivial solution when $x_0 \ge 0$.

The condition (6.10) is satisfied by a large number of functions h. Some examples and classes of h are given below, for instance, see Corollary 6.2 and Example 6.1.

Proof. Assume, on the contrary, that a solution $x \in AC[0,T]$ exists for all $T>0$. By establishing decaying bounds on the different terms arising in the weak formulation of the problem, we show that such a solution must be the trivial solution.

Multiplying both sides of the inequality in (6.9) by the test function φ defined in (6.1) with $\lambda > 2q' - 1$ and integrating, we obtain

$$\int_0^T \varphi(t) \left({}^C D_{0+}^{\mu} x\right)(t) dt \geq \int_0^T \varphi(t) \left(\int_0^t h(t-s) |x(s)|^q \, ds\right) dt. \tag{6.11}$$

We have from Lemma 6.5,

$$\int_0^T \varphi(t) \left({}^C D_{0+}^{\mu} x\right)(t) \, dt = \int_0^T x(t) \left(D_{T-}^{\mu} \varphi\right)(t) \, dt - x_0 \hat{G}_{0,\mu} T^{1-\mu}, \tag{6.12}$$

where $\hat{G}_{0,\mu} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+2)}$. Substituting (6.12) in (6.11) yields

$$J + x_0 \hat{G}_{0,\mu} T^{1-\mu} \leq \int_0^T x(t) \left(D_{T-}^{\mu} \varphi\right)(t) \, dt, \tag{6.13}$$

where

$$J := \int_0^T \varphi(t) \left(\int_0^t h(t-s) |x(s)|^q \, ds\right) dt. \tag{6.14}$$

To obtain a bound for the expression J, we rewrite J as

$$J = \int_0^T |x(s)|^q \left(\int_s^T h(t-s) \varphi(t) \, dt\right) ds = \int_0^T |x(s)|^q K(s) ds, \tag{6.15}$$

where

$$K(s) := \int_s^T h(t-s) \varphi(t) \, dt, \quad 0 \leq s < t \leq T. \tag{6.16}$$

By Hölder's inequality with $\frac{1}{q}+\frac{1}{q'}=1$, we obtain

$$\begin{aligned}\int_0^T x(t)\left(D_{T-}^{\mu}\varphi\right)(t)\,dt &\leq \int_0^T x(t)\,K^{\frac{1}{q}}(t)\,K^{-\frac{1}{q}}(t)\left(D_{T-}^{\mu}\varphi\right)(t)\,dt \\ &\leq \left(\int_0^T |x(t)|^q K(t)\,dt\right)^{\frac{1}{q}}\left(\int_0^T K^{-\frac{q'}{q}}(t)\left(D_{T-}^{\mu}\varphi\right)^{q'}(t)\,dt\right)^{\frac{1}{q'}}.\end{aligned}$$

Therefore (6.13) can be rewritten as

$$J\ +x_0\hat{G}_{0,\mu}T^{1-\mu}\leq J^{\ \frac{1}{q}}\left(\int_0^T K^{-\frac{q'}{q}}(t)\left(D_{T-}^{\mu}\varphi\right)^{q'}(t)\,dt\right)^{\frac{1}{q'}}. \qquad (6.17)$$

From the positivity of $J,\hat{G}_{0,\mu}$ and the nonnegativity of x_0, we get from (6.17),

$$J\ \leq J^{\ \frac{1}{q}}\left(\int_0^T K^{-\frac{q'}{q}}(t)\left(D_{T-}^{\mu}\varphi\right)^{q'}(t)\,dt\right)^{\frac{1}{q'}},$$

which implies that

$$J\ \leq\int_0^T K^{-\frac{q'}{q}}(t)\left(D_{T-}^{\mu}\varphi\right)^{q'}(t)\,dt. \qquad (6.18)$$

But, we see from Lemma 6.6 that

$$\begin{aligned}J &\leq \int_0^T K^{-\frac{q'}{q}}(t)\left(D_{T-}^{\mu}\varphi\right)^{q'}(t)\,dt=\int_0^T\left(\int_t^T h(s-t)\varphi(s)\,ds\right)^{1-q'}\left(D_{T-}^{\mu}\varphi\right)^{q'}(t)\,dt \\ &\leq \hat{C}_{\mu,q'}T^{1-q'}\int_0^T t^{-\mu q'}h^{1-q'}(t)dt. \qquad (6.19)\end{aligned}$$

It follows, from the assumption (6.10), that $x\equiv 0$ and this completes the proof. ∎

As a corollary of Theorem 6.1, we have the following result.

Corollary 6.1 *Let $0<\mu<1$ and h be a nonnegative function which is different* *from zero almost everywhere with* $t^{-\mu q'} h^{1-q'}(t) \in L^1_{loc}[0,\infty)$. *Assume that, for any* $T > 0$, *there are some positive constants* c *and*

$$\theta < \frac{1}{q-1} \tag{6.20}$$

such that

$$\int_0^T t^{-\mu q'} h^{1-q'}(t) dt \leq cT^{\theta}, \tag{6.21}$$

where $q' = \frac{q}{q-1}$. *Then, the problem* (6.9) *does not admit any global nontrivial solution when* $x_0 \geq 0$.

Proof. To prove this corollary it suffices to notice that the conditions (6.20) and (6.21) imply that the hypothesis (6.10) is fulfilled. Indeed, in virtue of (6.21), we have

$$0 \leq T^{1-q'} \int_0^T t^{-\mu q'} h^{1-q'}(t) dt \leq cT^{1-q'+\theta}.$$

We find from (6.20) that $1 - q' + \theta < 0$ and the condition (6.10) follows. ∎

The following corollary of Theorem 6.1 is concerned with a large class of kernels that appear in applications.

Corollary 6.2 *Let* $0 < \mu < 1$ *and* $q > 1$. *Suppose that* $h(t) \geq at^{-\nu}$, $t > 0$, *for some constant* $a > 0$, *where* $1 - q(1-\mu) < \nu < 2 + q(\mu - 1)$. *Then, the problem* (6.9) *does not admit a global nontrivial solution when* $x_0 \geq 0$.

Proof. It suffices to show that the function h satisfies (6.10). Indeed, $h(t) \geq$

$at^{-\nu}$, implies that $h^{1-q'}(t) \leq a^{1-q'} t^{\nu(q'-1)}$, $a > 0$, and

$$\begin{aligned}\int_0^T t^{-\mu q'} h^{1-q'}(t)dt &\leq a^{1-q'} \int_0^T t^{\nu(q'-1)-\mu q'} dt \\ &= \frac{a^{1-q'}}{\nu(q'-1)-\mu q'+1} T^{\nu(q'-1)-\mu q'+1},\end{aligned}$$

where $\nu(q'-1)-\mu q'+1 > 0$ as a consequence of $\nu > 1-q(1-\mu)$. Therefore,

$$T^{1-q'} \int_0^T t^{-\mu q'} h^{1-q'}(t)dt \leq \frac{a^{1-q'}}{\nu(q'-1)-\mu q'+1} T^{2-\nu+q'(\nu-\mu-1)}.$$

It follows from $\nu < 2+q(\mu-1)$ that $2-\nu+q'(\nu-\mu-1) < 0$ and so (6.10) is satisfied. ∎

Remark 6.1 *Corollary 6.2 can be looked at also as a consequence of Corollary 6.1 with*

$$\begin{aligned}c &= \frac{a^{1-q'}}{\nu(q'-1)-\mu q'+1}, \\ \theta &= \nu(q'-1)-\mu q'+1 = \frac{q(1-\mu)+\nu-1}{q-1}, \quad 0 \leq \mu < 1.\end{aligned}$$

It is clear from $1-q(1-\mu) < \nu < 2-q(1-\mu)$ *that* $0 < \theta < \frac{1}{q-1}$.

The next example treats a special case of the kernel h in Corollary 6.4.

Example 6.1 *Consider the problem*

$$\begin{cases} \left({}^C D^{\mu}_{0+} x\right)(t) \geq \left(I^{\beta_1}_{0+} |x(s)|^q\right)(t), \quad t > 0, \ 0 < \mu < 1, \ \beta_1 > 0, \ q > 1, \\ x(0) = x_0, \quad x_0 \geq 0. \end{cases} \tag{6.22}$$

This problem is a special case of the problem (6.9) *with*

$$h(t) = t^{\beta_1 - 1}, \quad 0 < \beta_1 < q\,(1-\mu)\,,\ 0 < \mu < 1.$$

Therefore, as a consequence of Corollary 6.2 with $h(t) = t^{-\nu}$, $\nu = 1 - \beta_1$, *the problem* (6.22) *does not admit a global nontrivial solution when* $x_0 \geq 0$.

Now, to investigate the effect of the first-order derivative and the fractional derivative of order $0 \leq \mu < 1$, (when combined together) we consider the problem (6.8) with $\sigma = 1$.

Theorem 6.2 *Let* $0 \leq \mu < 1$. *Suppose that* h *is a nonnegative function which is different from zero almost everywhere with* $h^{1-q'}$, $t^{-\mu q'} h^{1-q'}(t) \in L^1_{loc}[0,\infty)$ *and*

$$\lim_{T\to\infty} T^{1-q'} \left(T^{-q'} \int_0^T h^{1-q'}(t)dt + \int_0^T t^{-\mu q'} h^{1-q'}(t)dt \right) = 0, \tag{6.23}$$

where $q' = \frac{q}{q-1}$. *Then, the problem*

$$\begin{cases} x'(t) + \left({}^{C}D^{\mu}_{0+}x\right)(t) \geq \int_0^t h(t-s)\,|x(s)|^q\,ds,\ t>0,\ q>1, \\ x(0) = x_0,\ x_0 \in \mathbb{R}, \end{cases} \tag{6.24}$$

does not admit any global nontrivial solution when $x_0 \geq 0$.

Proof. Assume, on the contrary, that a solution $x \in AC[0,T]$ exists for all

$T > 0$. We have

$$\int_0^T \varphi(t)\, x'(t)dt + \int_0^T \varphi(t) \left({}^C D_{0+}^{\mu} x\right)(t)dt \geq \int_0^T \varphi(t) \left(\int_0^t h(t-s)\, |x(s)|^q\, ds\right) dt. \tag{6.25}$$

A simple integration by parts (a special case of Lemma 6.5), yields

$$\int_0^T \varphi(t)\, x'(t)dt = -\int_0^T x(t)\varphi'(t)\, dt - x_0.$$

Therefore,

$$J + x_0 \left(1 + \hat{G}_{0,\mu} T^{1-\mu}\right) \leq \int_0^T x(t)\left(-\varphi'(t)\right) dt + \int_0^T x(t) \left(D_{T-}^{\mu}\varphi\right)(t)\, dt, \tag{6.26}$$

where J is given in (6.14) and (6.15). We deduce after applying Hölder's inequality to the integrals in the right-hand side of (6.26),

$$J\ + x_0 \left(1 + \hat{G}_{0,\mu} T^{1-\mu}\right) \leq J^{\,\frac{1}{q}} \left(A^{\frac{1}{q'}} + B^{\frac{1}{q'}}\right), \tag{6.27}$$

where

$$A := \int_0^T K^{-\frac{q'}{q}}(t) \left(-\varphi'(t)\right)^{q'} dt, \quad B := \int_0^T K^{-\frac{q'}{q}}(t) \left(D_{T^-}^{\mu}\varphi(t)\right)^{q'} dt. \tag{6.28}$$

The inequality (6.27) leads to

$$J\ \leq J^{\,\frac{1}{q}} \left(A^{\frac{1}{q'}} + B^{\frac{1}{q'}}\right),$$

or

$$J \ \leq 2^{q'-1}\left(A+B\right). \tag{6.29}$$

The integral A is estimated by

$$\begin{aligned} A &= \int_0^T K^{-\frac{q'}{q}}(t)\left(-\varphi'(t)\right)^{q'} dt = \int_0^T \left(\int_t^T h(s-t)\varphi(s)\,ds\right)^{1-q'} \left(D^1_{T-}\varphi(t)\right)^{q'} dt \\ &\leq \hat{C}_{1,q'} T^{1-2q'} \int_0^T h^{1-q'}(t)dt, \ \text{(Lemma 6.6 with } \rho = 1). \end{aligned} \tag{6.30}$$

We know from the proof of Theorem 6.1 that

$$B \leq \hat{C}_{\mu,q'} T^{1-q'} \int_0^T t^{-\mu q'} h^{1-q'}(t)dt. \tag{6.31}$$

Substituting (6.30) and (6.31) in (6.29) we end up with

$$J \ \leq \hat{C}\left(T^{1-2q'}\int_0^T h^{1-q'}(t)dt + T^{1-q'}\int_0^T t^{-\mu q'} h^{1-q'}(t)dt\right), \tag{6.32}$$

where $\hat{C} = 2^{q'-1}\max\left\{\hat{C}_{1,q'}, \hat{C}_{\mu,q'}\right\}$. It follows from the assumption (6.23) that $x \equiv 0$ which leads to a contradiction since the solution is supposed to be nontrivial.

∎

In line with the Corollary 6.1 of Theorem 6.1, we can prove the following corollary of Theorem 6.2.

Corollary 6.3 *Let $0 \leq \mu < 1$ and suppose that h is a nonnegative function which is different from zero almost everywhere with $h^{1-q'}$, $t^{-\mu q'} h^{1-q'}(t) \in L^1_{loc}[0,\infty)$.*

Assume that, for any $T > 0$, there are some positive constants c_1, c_2,

$$\theta_1 < \frac{q+1}{q-1} \text{ and } \theta_2 < \frac{1}{q-1} \tag{6.33}$$

such that

$$\int_0^T h^{1-q'}(t)dt \le c_1 T^{\theta_1} \text{ and } \int_0^T t^{-\mu q'} h^{1-q'}(t)dt \le c_2 T^{\theta_2}, \tag{6.34}$$

where $q' = \frac{q}{q-1}$. *Then, the problem* (6.24) *does not admit any global nontrivial solution when* $x_0 \ge 0$.

Proof. It enough to show that the conditions (6.33) and (6.34) imply that (6.23) is satisfied. Since

$$0 \le T^{1-q'} \left(T^{-q'} \int_0^T h^{1-q'}(t)dt + \int_0^T t^{-\mu q'} h^{1-q'}(t)dt \right) \le c_1 T^{1-2q'+\theta_1} + c_2 T^{1-q'+\theta_2},$$

we conclude from (6.33) that $1 - 2q' + \theta_1 < 0$, $1 - q' + \theta_2 < 0$ and the condition (6.23) is fulfilled. ∎

Corollary 6.4 *Let* $0 \le \mu < 1$ *and* $q > 1$. *Suppose that* $h(t) \ge a t^{-\nu}$, $t > 0$, *for some constant* $a > 0$, *where* $1 - q(1-\mu) < \nu < 2 + q(\mu - 1)$. *Then, the problem* (6.24) *does not admit a global nontrivial solution when* $x_0 \ge 0$.

Proof. Showing that h satisfies (6.23) suffices to prove our assertion. Since

$h(t) \geq at^{-\nu}$, $a > 0$, it is easy to see that

$$\int_0^T h^{1-q'}(t)dt \leq a^{1-q'} \int_0^T t^{\nu(q'-1)}dt = \frac{a^{1-q'}}{\nu\left(q'-1\right)+1}T^{\nu(q'-1)+1},$$

$$\int_0^T t^{-\mu q'} h^{1-q'}(t)dt \leq \frac{a^{1-q'}}{\nu\left(q'-1\right)-\mu q'+1}T^{\nu(q'-1)-\mu q'+1},$$

and

$$T^{1-2q'} \int_0^T h^{1-q'}(t)dt + T^{1-q'} \int_0^T t^{-\mu q'} h^{1-q'}(t)dt \leq \frac{a^{1-q'}T^{\sigma_1}}{\nu\left(q'-1\right)+1} + \frac{a^{1-q'}T^{\sigma_2}}{\nu\left(q'-1\right)-\mu q'+1},$$

where

$$\sigma_1 = 2-\nu+q'\left(\nu-2\right),\ \sigma_2 = 2-\nu+q'\left(\nu-\mu-1\right).$$

Now, the assumption $\nu < 2+q\left(\mu-1\right)$ implies that both σ_1 and σ_2 are negative. The proof is complete. ▮

Remark 6.2 *The presence of the first order derivative of the solution does not prevent the nonexistence of nontrivial global solutions when* $0 \leq \mu < 1$*. It does not effect the range of values of the exponent* ν *in the case of power-type kernel in corollaries 6.2 and 6.4 neither. Indeed, the fractional derivative plays the role of a damping term in the problems treated in Theorem 6.2 and its corollaries.*

Remark 6.3 *With*

$$\begin{aligned}
c_1 &= \frac{a^{1-q'}}{\nu\left(q'-1\right)+1}, \; c_2 = \frac{a^{1-q'}}{\nu\left(q'-1\right)-\mu q'+1}, \\
\theta_1 &= \nu\left(q'-1\right)+1 = \frac{q+\nu-1}{q-1}, \\
\theta_2 &= \nu\left(q'-1\right)-\mu q'+1 = \frac{q\left(1-\mu\right)+\nu-1}{q-1}, \quad 0 \leq \mu < 1,
\end{aligned}$$

Corollary 6.4 can be derived from Corollary 6.3.

6.2.2 Case $1 \leq \mu < 2,\ \beta = 1$

This subsection is devoted to the study of the nonexistence of nontrivial global solutions for the problem (6.7) in the space $AC^2\left[0,T\right]$ for all $T > 0$ with $1 \leq \mu < 2$, $\beta = 1$. For $1 < \mu < 2$ and $\sigma = 0$, we have

$$\begin{cases} \left({}^C D_{0+}^{\mu} x\right)(t) \geq \int_0^t h(t-s)\left|x(s)\right|^q ds, \; t > 0, \; q > 1, \; 1 < \mu < 2, \\ x(0) = x_0, \; x'(0) = x_1, \; x_0, x_1 \in \mathbb{R}. \end{cases} \tag{6.35}$$

Theorem 6.3 *Let $1 < \mu < 2$. Assume that h is a nonnegative function which is different from zero almost everywhere with $t^{(1-\mu)q'} h^{1-q'} \in L^1_{loc}\left[0,\infty\right)$ and*

$$\lim_{T\to\infty} T^{1-2q'} \int_0^T t^{(1-\mu)q'} h^{1-q'}(t)dt = 0, \tag{6.36}$$

where $q' = \frac{q}{q-1}$. Then, the initial value problem (6.35) does not admit any global nontrivial solution when $x_0, x_1 \geq 0$.

Proof. For the sake of contradiction, suppose that a (nontrivial) solution $x \in AC^2[0,T]$ exists for all $T > 0$. Then, as in the proof of Theorem 6.1, we obtain from the weak formulation of the problem

$$J + x_0 \hat{G}_{1,\mu} T^{1-\mu} + x_1 \hat{G}_{0,\mu} T^{2-\mu} \leq J^{\frac{1}{q}} \left(\int_0^T K^{-\frac{q'}{q}}(t) \left(D_{T^-}^{\mu} \varphi \right)^{q'}(t)\, dt \right)^{\frac{1}{q'}},$$

where J is as in (6.14). Accordingly, for $1 \leq \mu < 2$, from Lemma 6.6 with $1 \leq \rho = \mu < 2$, we have

$$\int_0^T K^{-\frac{q'}{q}}(t) \left(D_{T^-}^{\mu} \varphi \right)^{q'}(t)\, dt \leq \hat{C}_{\mu,q'} T^{1-2q'} \int_0^T t^{(1-\mu)q'} h^{1-q'}(t) dt.$$

By the assumption (6.36), we get $x \equiv 0$ and the proof is complete. ∎

Corollary 6.5 *Let* $1 < \mu < 2$. *Suppose that* $h(t) \geq at^{-\nu}$, $t > 0$, *for some constant* $a > 0$, *where* $1 - q(2-\mu) < \nu < 2 + q(\mu - 1)$. *Then, the problem* (6.35) *does not admit a global nontrivial solution when* $x_0, x_1 \geq 0$.

Proof. It suffices to notice that the kernel h satisfies the hypotheses of Theorem 6.3. Clearly,

$$\int_0^T t^{(1-\mu)q'} h^{1-q'}(t) dt \leq a^{1-q'} \int_0^T t^{\nu(q'-1)+(1-\mu)q'} dt = bT^{\eta},$$

where

$$\begin{aligned} b &= \frac{a^{1-q'}}{\nu\left(q'-1\right)+\left(1-\mu\right)q'+1}, \\ \eta &= \nu\left(q'-1\right)+\left(1-\mu\right)q'+1 = \frac{q\left(2-\mu\right)+\nu-1}{q-1}. \end{aligned}$$

Since

$$1-2q'+\eta = 2-\nu+q'\left(\nu-1-\mu\right) = \frac{q\left(1-\mu\right)+\nu-2}{q-1} < 0,$$

then (6.36) is satisfied. ∎

For the case $1 \leq \mu < 2$, $\beta = \sigma = 1$, we have the main theorem.

Theorem 6.4 *Let $1 \leq \mu < 2$. Assume that h is a nonnegative function which is different from zero almost everywhere with $h^{1-q'}$, $t^{(1-\mu)q'}h^{1-q'}(t) \in L^1_{loc}[0,\infty)$ and*

$$\lim_{T\to\infty} T^{1-2q'}\left(\int_0^T h^{1-q'}(t)dt + \int_0^T t^{(1-\mu)q'}h^{1-q'}(t)dt\right) = 0, \qquad (6.37)$$

where $q' = \frac{q}{q-1}$. Then, the initial value problem

$$\begin{cases} x'(t) + \left({}^{C}D^{\mu}_{0+}x\right)(t) \geq \int_0^t h(t-s)\left|x(s)\right|^q ds,\ t>0,\ q>1, \\ x(0) = x_0,\ x'(0) = x_1,\ x_0, x_1 \in \mathbb{R}, \end{cases} \qquad (6.38)$$

does not admit any global nontrivial solution when $x_0, x_1 \geq 0$.

Proof. Assume, on the contrary, that a solution $x \in AC^2\,[0,T]$ exists for all $T > 0$. Proceeding analogously to the proof of Theorem 6.2 gives

$$J + x_0\left(1 + \hat{G}_{1,\mu}T^{1-\mu}\right) + x_1\hat{G}_{0,\mu}T^{2-\mu} \leq J^{\frac{1}{q}}\left(A^{\frac{1}{q'}} + B^{\frac{1}{q'}}\right),$$

where J, A and B are as in (6.14) and (6.28). From Lemma 6.6 with $\rho = 1$ and $1 \leq \rho = \mu < 2$,

$$\begin{aligned}
A &\leq \hat{C}_{1,q'}T^{1-2q'}\int_0^T h^{1-q'}(t)dt,\\
B &= \int_0^T\left(\int_t^T h(s-t)\varphi\left(s\right)ds\right)^{1-q'}\left(D^{\mu}_{T^-}\varphi\right)^{q'}(t)\,dt\\
&\leq \hat{C}_{\mu,q'}T^{1-2q'}\int_0^T t^{(1-\mu)q'}h^{1-q'}(t)dt,
\end{aligned}$$

respectively. The assumption (6.37) allows us to conclude. ∎

For kernels satisfying $h(t) \geq t^{-\nu}$, Theorem 6.4 implies the corollary below.

Corollary 6.6 *Let* $1 \leq \mu < 2$. *Suppose that* $h(t) \geq at^{-\nu}$, $t > 0$, *for some constant* $a > 0$, *where* $1 - q\left(2-\mu\right) < \nu < 2$. *Then, the problem* (6.38) *does not admit a global nontrivial solution when* x_0, $x_1 \geq 0$.

Proof. It suffices to notice that the kernel h satisfies the hypotheses of Theorem 6.4. Indeed,

$$\int_0^T h^{1-q'}(t)dt \leq a^{1-q'}\int_0^T t^{\nu(q'-1)}dt = b_2T^{\eta_1},$$

$$\int_0^T t^{(1-\mu)q'}h^{1-q'}(t)dt \leq a^{1-q'}\int_0^T t^{\nu(q'-1)+(1-\mu)q'}dt = b_3T^{\eta_2},$$

where

$$
\begin{aligned}
b_2 &= \frac{a^{1-q'}}{\nu\,(q'-1)+1},\ b_3 = \frac{a^{1-q'}}{\nu\,(q'-1)+(1-\mu)\,q'+1}, \\
\eta_1 &= \nu\,(q'-1)+1 = \frac{q+\nu-1}{q-1}, \\
\eta_2 &= \nu\,(q'-1)+(1-\mu)\,q'+1 = \frac{q\,(2-\mu)+\nu-1}{q-1}.
\end{aligned}
$$

Moreover, it is easy to check that $\eta_1, \eta_2 > 0$ and $1-2q'+\eta_1, 1-2q'+\eta_2 < 0$. ▮

Remark 6.4 *We can get the same results of Theorem 6.4 and Corollary 6.6 with less restrictive conditions on the initial data. Instead of requiring x_0 and x_1 to be both nonnegative, it is enough to require $a_0x_0 + a_1x_1 \geq 0$ for some positive constants a_0 and a_1. Indeed, a_0 and a_1 can be expressed in terms of the constants $T, \hat{G}_{0,\mu}$ and $\hat{G}_{1,\mu}$.*

Remark 6.5 *Note that $\nu < 2 + q\,(\mu-1)$ in Corollary 6.5 while $\nu < 2 \leq 2 + q\,(\mu-1)\,,\ 1 \leq \mu < 2,$ in Corollary 6.6. We observe that the first derivative plays the role of a damping term when the order of the fractional derivative is $1 < \mu < 2$.*

In the next subsection, we consider the case of two fractional derivatives of general order.

6.2.3 Case $n-1 \leq \beta < \mu < n$

We return now to the initial value problem (6.7) and establish the following general theorem.

Theorem 6.5 *Let $n-1 \leq \beta < \mu < n$ and h be a nonnegative function which is different from zero almost everywhere. Suppose that $\left(t^{(n-\mu-1)q'} + \sigma^{q'} t^{(n-\beta-1)q'}\right) h^{1-q'}(t) \in L^1_{loc}[0,\infty)$ and*

$$\lim_{T\to\infty} T^{1-nq'} \left(\int_0^T \left(t^{(n-\mu-1)q'} + \sigma^{q'} t^{(n-\beta-1)q'} \right) h^{1-q'}(t) dt \right) = 0, \tag{6.39}$$

where $q' = \frac{q}{q-1}$. Then, the problem (6.7) does not admit any global nontrivial solution when $a_0x_0 + a_1x_1, ... + a_{n-1}x_{n-1} \geq 0$ for some positive constants $a_0, a_1, ..., a_{n-1}$ to be determined.

Proof. As in the proof of Theorem 6.1, we have here from Lemma 6.5,

$$\int_0^T \varphi(t) \left({}^C D^{\mu}_{0+} x\right)(t)\, dt = \int_0^T x(t) \left(D^{\mu}_{T-}\varphi\right)(t)\, dt - \sum_{j=0}^{n-1} \hat{G}_{j,\mu} T^{n-\mu-j} x_{n-1-j}, \tag{6.40}$$

$$\int_0^T \varphi(t) \left({}^C D^{\beta}_{0+} x\right)(t)\, dt = \int_0^T x(t) \left(D^{\beta}_{T-}\varphi\right)(t)\, dt - \sum_{j=0}^{n-1} \hat{G}_{j,\beta} T^{n-\beta-j} x_{n-1-j}. \tag{6.41}$$

Hence,

$$\begin{aligned} &J + \sum_{j=0}^{n-1} \left(\hat{G}_{j,\mu} T^{n-\mu-j} + \sigma \hat{G}_{j,\beta} T^{n-\beta-j} \right) x_{n-1-j} \\ &\leq J^{\frac{1}{q}} \left(\left(\int_0^T K^{-\frac{q'}{q}} \left(D^{\mu}_{T-}\varphi\right)^{q'} dt \right)^{\frac{1}{q'}} + \sigma \left(\int_0^T K^{-\frac{q'}{q}} \left(D^{\beta}_{T-}\varphi\right)^{q'} dt \right)^{\frac{1}{q'}} \right), \end{aligned}$$

which leads to

$$J \leq 2^{q'-1} \left(\int_0^T K^{-\frac{q'}{q}} \left(D^{\mu}_{T-}\varphi\right)^{q'} dt + \sigma^{q'} \int_0^T K^{-\frac{q'}{q}} \left(D^{\beta}_{T-}\varphi\right)^{q'} dt \right). \tag{6.42}$$

We have from Lemma 6.6,

$$\int_0^T K^{-\frac{q'}{q}}(t)\left|D^{\mu}_{T^-}\varphi(t)\right|^{q'}(t)\,dt \le \hat{C}_{\mu,q'}T^{1-nq'}\int_0^T t^{(n-\mu-1)q'}h^{1-q'}(t)dt, \tag{6.43}$$

$$\int_0^T K^{-\frac{q'}{q}}(t)\left|D^{\beta}_{T^-}\varphi\right|^{q'}(t)\,dt \le \hat{C}_{\beta,q'}T^{1-nq'}\int_0^T t^{(n-\beta-1)q'}h^{1-q'}(t)dt. \tag{6.44}$$

Using (6.43) and (6.44) in (6.42) and considering the condition (6.39) complete the proof. ∎

Corollary 6.7 *Let $n-1 \le \beta < \mu < n$ and $q > 1$. Suppose that $h(t) \ge at^{-\nu}$, $t > 0$, for some constant $a > 0$, where $1-q(n-\mu) < \nu < 2+q(\beta-1)$. Then, the problem (6.7) does not admit a global nontrivial solution when $a_0x_0 + a_1x_1, ... + a_{n-1}x_{n-1} \ge 0$ for some positive constants $a_0, a_1, ..., a_{n-1}$ to be determined.*

Proof. Here

$$\begin{aligned} &T^{1-nq'}\left(\int_0^T \left(t^{(n-\mu-1)q'} + \sigma^{q'}t^{(n-\beta-1)q'}\right)h^{1-q'}(t)dt\right) \\ &\le \frac{a^{1-q'}T^{2-\nu+q'(\nu-\mu-1)}}{q'(\nu+n-\mu-1)-\nu+1} + \frac{a^{1-q'}\sigma^{q'}T^{2-\nu+q'(\nu-\beta-1)}}{q'(\nu+n-\beta-1)-\nu+1}. \end{aligned}$$

Therefore, (6.39) is satisfied when $1-q(n-\mu) < \nu < 2+q(\beta-1)$. ∎

Note that the upper bound of the exponent ν is determined in terms of the smaller derivative β.

6.3 Problems with Riemann-Liouville Fractional Derivatives

In this section, we consider the initial value problem

$$\begin{cases} \left(D_{0^+}^{\mu}x\right)(t) + \sigma(D_{0^+}^{\beta}x)(t) \geq \int_0^t h(t-s)\left|x(s)\right|^q ds, \quad t>0,\ q>1, \\ \left(I_{0^+}^{1-\mu}x\right)(0^+) = c_0, \quad c_0 \in \mathbb{R}, \end{cases} \tag{6.45}$$

where $0 \leq \beta < \mu < 1$ and $\sigma = 0, 1$.

A nonexistence result of nontrivial global solutions $x \in C_{1-\mu}^{\mu}[0,T]$ for all $0 < T \leq \infty$, is investigated. Here the space $C_{1-\mu}^{\mu}[0,T]$ is the weighted space of continuous functions defined in (4.65).

Theorem 6.6 *Let $0 \leq \beta < \mu < 1$ and h be a nonnegative function which is different from zero almost everywhere. Assume that $\left(t^{-\mu q'} + \sigma^{q'} t^{-\beta q'}\right) h^{1-q'}(t) \in L_{loc}^{1}[0,\infty)$ and*

$$\lim_{T\to\infty} T^{1-q'}\left(\int_0^T t^{-\mu q'} h^{1-q'}(t)dt + \sigma^{q'}\int_0^T t^{-\beta q'} h^{1-q'}(t)dt\right) = 0, \tag{6.46}$$

where $q' = \frac{q}{q-1}$. Then, the problem (6.45) does not admit any global nontrivial solution when $c_0 \geq 0$.

Proof. Suppose, contrary to our claim, that a solution $x \in C_{1-\mu}^{\mu}[0,T]$ exists for all $T > 0$. Multiplying both sides of the inequality in (6.45) by the test function

φ defined in (6.1) with $\lambda > 2q' - 1$ and integrating, we obtain

$$J \leq \int_0^T \varphi(t) \left(D_{0^+}^{\mu} x\right)(t)\, dt + \sigma \int_0^T \varphi(t) \left(D_{0^+}^{\beta} x\right)(t)\, dt, \tag{6.47}$$

where J is given in (6.14). An integration by parts for each integral in right-hand side of (6.47) gives

$$\int_0^T \varphi(t) \left(D_{0^+}^{\mu} x\right)(t)\, dt = \left[\varphi(t) \left(I_{0^+}^{1-\mu} x\right)(t)\right]_{t=0}^{T} - \int_0^T \varphi'(t) \left(I_{0^+}^{1-\mu} x\right)(t)\, dt,$$

and

$$\int_0^T \varphi(t) \left(D_{0^+}^{\beta} x\right)(t)\, dt = \left[\varphi(t) \left(I_{0^+}^{1-\beta} x\right)(t)\right]_{t=0}^{T} - \int_0^T \varphi'(t) \left(I_{0^+}^{1-\beta} x\right)(t)\, dt.$$

As $\varphi(T) = 0$, $\varphi(0) = 1$ and $\left(I_{0^+}^{1-\mu} x\right)(0^+) = c_0$, we find

$$\int_0^T \varphi(t) \left(D_{0^+}^{\mu} x\right)(t)\, dt = -c_0 - \int_0^T \varphi'(t) \left(I_{0^+}^{1-\mu} x\right)(t)\, dt.$$

Also, since $I_{0^+}^{1-\beta} x = I_{0^+}^{\mu-\beta} I_{0^+}^{1-\mu} x$ and $I_{0^+}^{1-\mu} x \in C[0,T]$, we see from Lemma 3.15 that $\left(I_{0^+}^{1-\beta} x\right)(0^+) = 0$. Hence

$$\int_0^T \varphi(t) \left(D_{0^+}^{\beta} x\right)(t)\, dt = -\int_0^T \varphi'(t) \left(I_{0^+}^{1-\beta} x\right)(t)\, dt,$$

and (6.47) becomes

$$J \leq -c_0 - \int_0^T \varphi'(t) \left(I_{0^+}^{1-\mu} x\right)(t)\, dt - \sigma \int_0^T \varphi'(t) \left(I_{0^+}^{1-\beta} x\right)(t)\, dt. \tag{6.48}$$

Having in mind that $c_0 \geq 0$ and φ' is negative, we entail that

$$\begin{aligned} J &\leq \int_0^T (-\varphi'(t)) \left(I_{0^+}^{1-\mu} x\right)(t)\, dt + \sigma \int_0^T (-\varphi'(t)) \left(I_{0^+}^{1-\beta} x\right)(t)\, dt \\ &\leq \int_0^T (-\varphi'(t)) \left(I_{0^+}^{1-\mu} |x|\right)(t)\, dt + \sigma \int_0^T (-\varphi'(t)) \left(I_{0^+}^{1-\beta} |x|\right)(t)\, dt. \end{aligned} \tag{6.49}$$

Applying Lemma 3.9 to each integral in the right-hand side of (6.49), we obtain

$$J \leq \int_0^T |x(t)| \left(I_{T^-}^{1-\mu} (-\varphi')\right)(t)\, dt + \sigma \int_0^T |x(t)| \left(I_{T^-}^{1-\beta} (-\varphi')\right)(t)\, dt. \tag{6.50}$$

Next, we insert $K^{\frac{1}{q}}(t) K^{-\frac{1}{q}}(t)$ inside each integral in the right-hand side of (6.50) and apply Hölder's inequality,

$$J \leq J^{\frac{1}{q}} \left(\int_0^T K^{1-q'}(t) \left(I_{T^-}^{1-\mu} (-\varphi')\right)^{q'}(t)\, dt\right)^{\frac{1}{q'}} + \sigma \left(\int_0^T K^{\frac{-q'}{q}}(t) \left(I_{T^-}^{1-\beta} (-\varphi')\right)^{q'}(t)\, dt\right)^{\frac{1}{q'}}$$

In view of the fact

$$D_{T^-}^{\mu} \varphi = {}^{C}D_{T^-}^{\mu} \varphi = -I_{T^-}^{1-\mu} \varphi',$$

we find that

$$J \leq 2^{q'-1} \left(\int_0^T K^{1-q'}(t) \left(D_{T^-}^{\mu} \varphi\right)^{q'}(t)\, dt + \sigma^{q'} \int_0^T K^{\frac{-q'}{q}}(t) \left(D_{T^-}^{\beta} \varphi\right)^{q'}(t)\, dt\right).$$

Using Lemma 6.6, we get

$$J \leq \hat{C}_1 T^{1-q'} \left(\int_0^T t^{-\mu q'} h^{1-q'}(t) dt + \sigma^{q'} \int_0^T t^{-\beta q'} h^{1-q'}(t) dt \right), \tag{6.51}$$

where $\hat{C}_1 = 2^{q'-1} \max \left\{ \hat{C}_{\mu,q'}, \hat{C}_{\beta,q'} \right\}$. The assumption (6.46) leads to a contradiction since the solution is supposed to be nontrivial. ∎

Remark 6.6 *If we consider* $h(t) \geq at^{-\nu}$, $a > 0$, $1 - q(1-\mu) < \nu < 2 + q(\beta - 1)$ *and* $0 \leq \beta < \mu < 1$, *then, the problem* (6.45) *does not admit a global nontrivial solution when* $c_0 \geq 0$. *Further,*

$$\int_0^t h(t-s) |x(s)|^q \, ds = \left(I_{0+}^{\beta_1} |x(s)|^q \right)(t), \;\; q(1-\beta) - 1 < \beta_1 < q(1-\mu),$$

can be considered here as a special case of the problem (6.45).

Remark 6.7 *We believe that it is possible to extend the result of Theorem 6.6 to any arbitrary orders* $n - 1 \leq \beta < \mu < n$. *This will be the subject of future investigations.*

CHAPTER 7

CONCLUSION AND FUTURE WORK

In this dissertation, we considered some classes of fractional integro-differential equations with the Caputo or Riemann-Liouville derivatives in the left-hand sides and right-hand sides depending on the solution, its fractional derivatives as well as an integral of a kernel involving the solution or its fractional derivatives. We assumed the continuity of the nonlinearities and the boundedness of these nonlinearities by sums or products of continuous functions of time, in certain Lebesgue spaces, and nondecreasing functions of the states. Under these nonlinear growth conditions on the nonlinearities, we treated initial value problems for which, in general, solutions cannot be found explicitly. We found that their solutions, under certain conditions, behave like the solutions of the associated linear fractional differential equations with zero right-hand sides.

In addition to the singularity of the fractional operators, one of the challenges that we have faced is the validity of most of the estimates and inequalities at or near zero. We solved this issue by using the desingularization techniques to have the appropriate estimates. By going around this obstruction, we improved several results in the literature, concerning the integer and noninteger orders, where the authors avoided working near zero.

In the study of the nonexistence of nontrivial global solutions, different fractional damping for a class of fractional integro-differential problem have been considered. Unlike existing results, the source term is a convolution and therefore nonlocal in time. Polynomials and fractional integrals of polynomials become special cases. The weak formulations for the problem with an appropriate test function and several appropriate estimations have been used. Reasonable conditions ensuring nonexistence of global solutions are determined. Singular kernels illustrating interesting cases in applications which correspond to fractional integrals (of polynomials of the state) have been provided and discussed. The results can be used to analyze many models with nonlocal source terms. The examples we have provided represent a wide class of kernels and clearly illustrate, in suitable manner, the results.

Our study opens many possible directions for future works. We leave the consideration of other kinds of derivatives to future investigations. Our approaches can be generalized to problems involving the p-Laplacian (or, generally φ-Laplacian) operators. We plan to extend our results to other systems of fractional integro-differential equations. Our approaches can be used to study the long-time behavior and nonexistence of solutions for fractional differential equations with different distributed delays and arguments such as piecewise constant arguments.

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Vitae

- Name: Ahmad Mahadi Mugbil Ahmad
- Nationality: Yemeni
- Date of Birth: 15 July 1975
- Email: *g200704810@kfupm.edu.sa, alati2009@gmail.com*
- Permenant Address: Sanaa Street, Thamar, Yemen

Academic Qualifications:

- Doctor of Philosophy (Ph.D.) in applied mathematics (2011-2016), King Fahd University of Petroleum and Minerals (KFUPM), KSA
- Master of Science (M.Sc.) in applied mathematics (2008-2011), King Fahd University of Petroleum and Minerals (KFUPM), KSA
- Bachelor of Education (B.Ed.) in Mathematics (1997-2001), Thamar University, Yemen

Publications:

- Ahmad M. Ahmad and Ashfaque H. Bokhari and F. D. Zaman, Symmetries, Conservation Laws, and Wave Equation on the Milne Metric, Journal of Applied Mathematics, vol. 2012, Article ID 153817, 7 pages, 2012.

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Seminars:

- Oscillation for a differential equation of non-integer order, Math. Stat., departmental seminar, 05/05/2013.
- On a new generalized fractional derivative, Math. Stat., departmental seminar, 15/12/2013.

Workshops:

- Inquiry-Based Student-Centered Instruction, Apr. 13, 2016, Teaching and Learning Center (TLC), KFUPM.
- Thinking Over: Designing Experiments of Classroom Transformation, Feb. 29, 2016, TLC, KFUPM.
- Experiences in creating excitement in the classroom through active learning techniques, Feb. 11, 2016, TLC, KFUPM.
- Blackboard 9.1: Advanced Workshop I: Assignments, Grade Center, and Tests, Nov. 06, 2014, e-Learning Center (ELC), KFUPM.
- Introduction to Blackboard 9.1, Sep. 18, 2013, ELC, KFUPM.